Types as Weak ω-Groupoids

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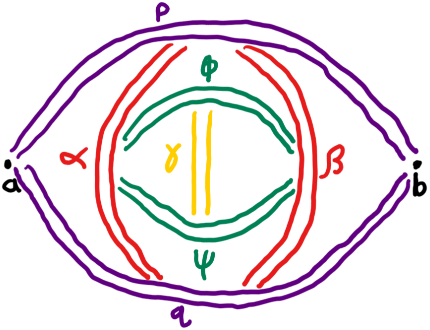
School and Workshop on Univalent Mathematics

Types

- What information does a type in our theory carry?
- Elements: $a, b : A$
- Equalities: $p, q : a =_A b$
- More equalities: $\alpha, \beta : p =_{a=b} q$
- And so on: $\phi, \psi : \alpha =_{p=q} \beta$
- And so forth: $\gamma : \phi =_{\alpha=\beta} \psi$
- With path induction: To prove for all γ, it suffices to assume...



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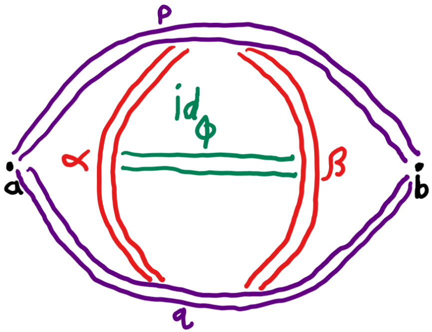

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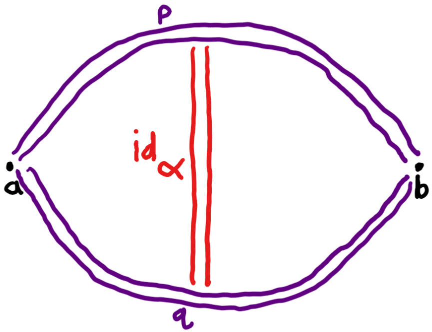

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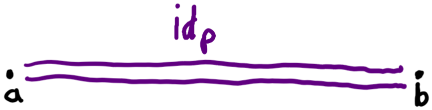

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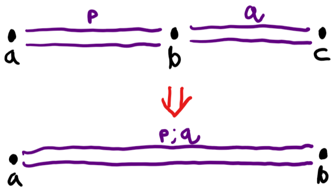
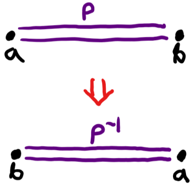
- Path induction gives us nice things:
- Composition. Symmetry.





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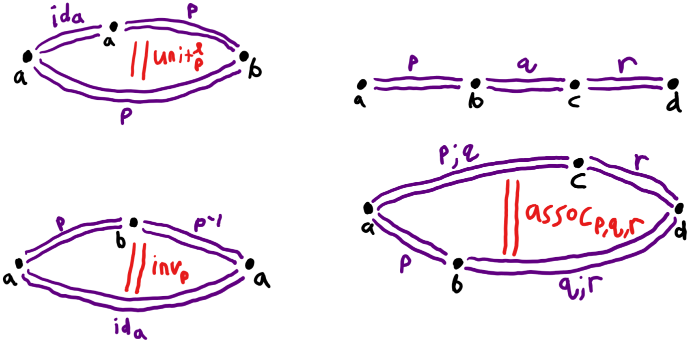
- Units. Associativity. Inverses.



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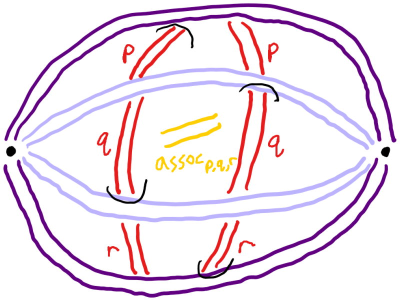

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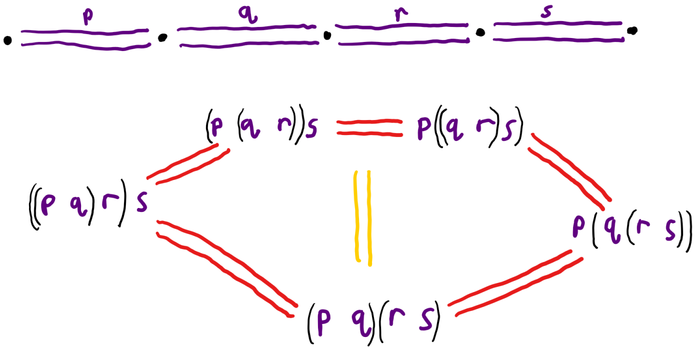
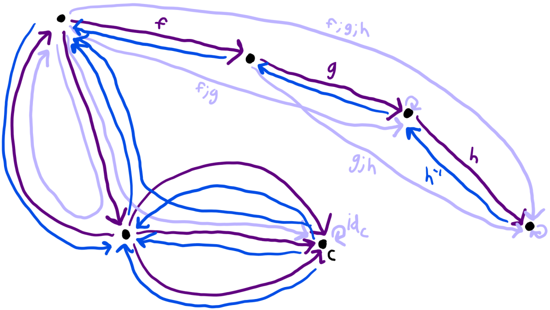
- Path induction gives us nice things:
- Composition. Symmetry.
- Units. Associativity. Inverses.
- At every level, but only up to higher cells.
- Higher order properties... What is this structure?



Composition Structures

- Sets have objects in X_0 (like $a, b : A$)
- Graphs are sets with arrows in X_1 (like $\phi : a =_A b$)
- Categories are graphs with composition, units, associativity (strict)
- Groupoids are categories with inverses (strict)

$$X_0 \underset{t}{\overset{s}{\leftleftarrows}} X_1$$



Composition Structures

- 0-Graphs have objects in X_0
- 1-Graphs are 0-graphs with arrows in X_1
- 1-Categories are 1-graphs with composition, units, associativity
- 1-Groupoids are 1-categories with inverses

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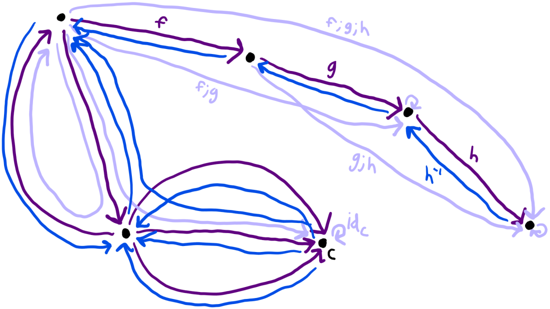
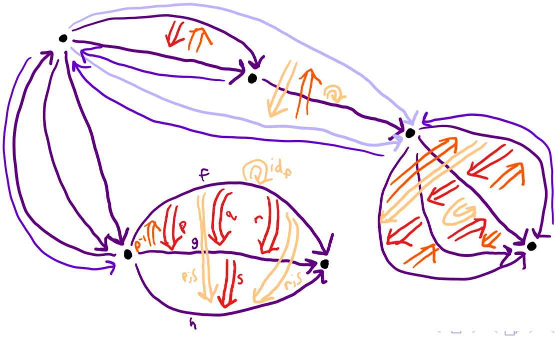


Composition Structures

- 1-Graphs have X_0, arrows in X_1
- 2-Graphs are 1-graphs with arrows in X_2
- 2-Categories are 2-graphs with composition, units, associativity
- 2-Groupoids are 2-categories with inverses

$$X_0 \underset{t}{\overset{s}{\leftleftarrows}} X_1 \underset{t}{\overset{s}{\leftleftarrows}} X_2$$



Composition Structures

- 2-Graphs have X_0, X_1, arrows in X_2
- 3-Graphs are 2-graphs with arrows in X_3
- 3-Categories are 3-graphs with composition, units, associativity
- 3-Groupoids are 3-categories with inverses

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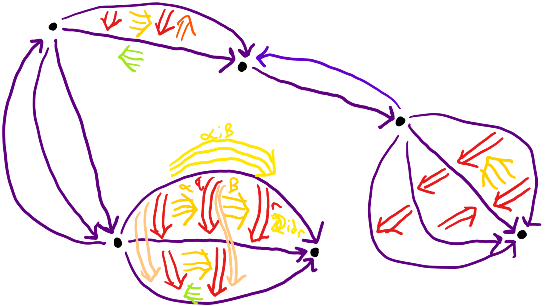
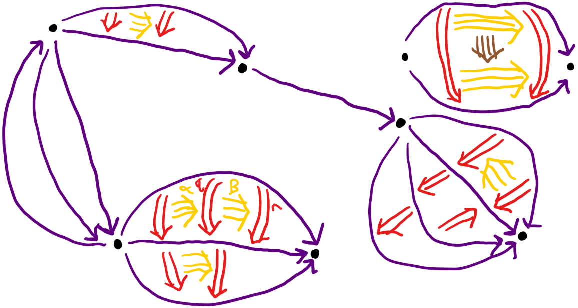


- n-Graphs have $X_0, \cdots, X_{n-1}$, arrows in X_n
- $(n+1)$-Graphs are n-graphs with arrows in X_{n+1}
- n-Categories are n-graphs with composition, units, associativity
- n-Groupoids are n-categories with inverses

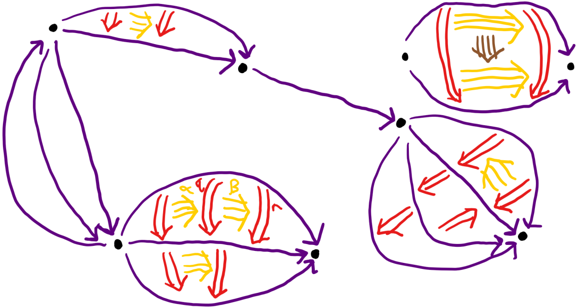
$$X_0 \underset{t}{\overset{s}{\leftleftarrows}} X_1 \underset{t}{\overset{s}{\leftleftarrows}} X_2 \underset{t}{\overset{s}{\leftleftarrows}} X_3 \underset{t}{\overset{s}{\leftleftarrows}} \cdots \underset{t}{\overset{s}{\leftleftarrows}} X_n$$



Composition Structures

- ω-Graphs have $X_0, X_1, X_2, \cdots$
- ω-Graphs are called globular sets, arrows in X_n are n-cells
- ω-Categories are ω-graphs with composition, units, associativity
- ω-Groupoids are ω-categories with inverses

$$X_0 \overset{s}{\underset{t}{\leftleftarrows}} X_1 \overset{s}{\underset{t}{\leftleftarrows}} X_2 \overset{s}{\underset{t}{\leftleftarrows}} X_3 \overset{s}{\underset{t}{\leftleftarrows}} \cdots \overset{s}{\underset{t}{\leftleftarrows}} X_n \overset{s}{\underset{t}{\leftleftarrows}} \cdots$$



Composable Shapes

- Let X be a globular set

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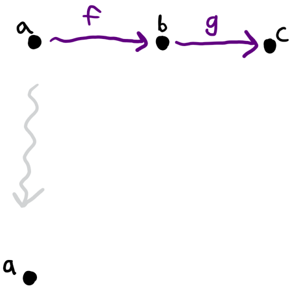
- What should we be able to compose, and into what?
- Free unbiased pasting diagrams in dimension 1



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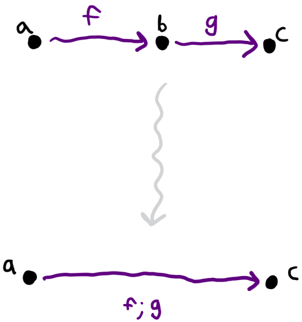

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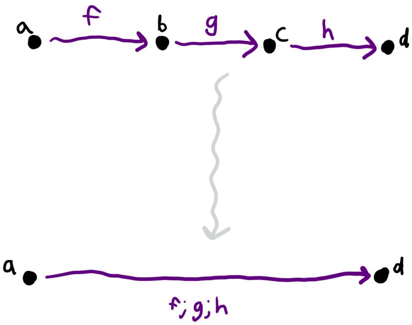

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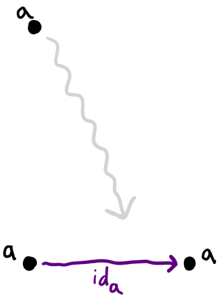

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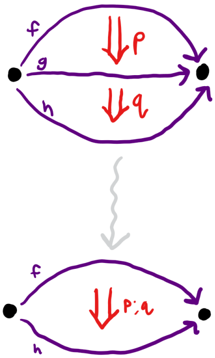

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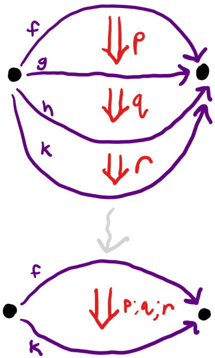

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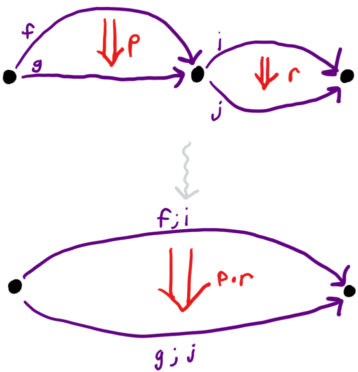

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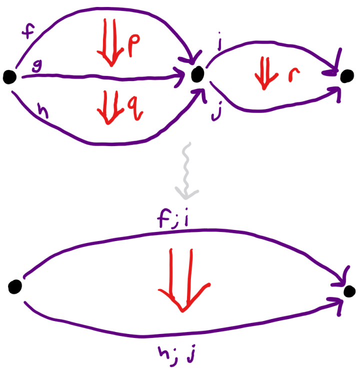

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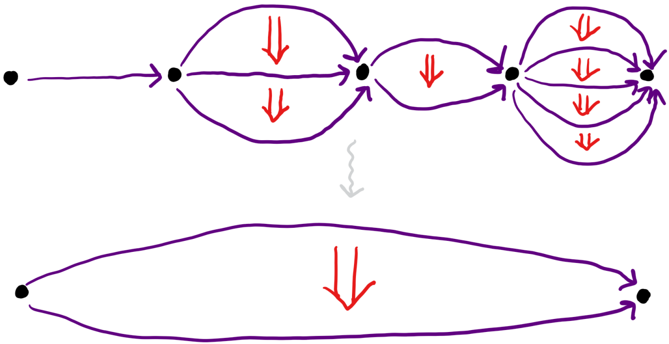

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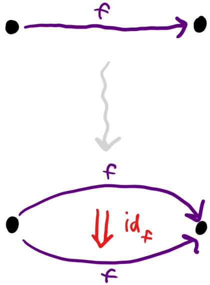

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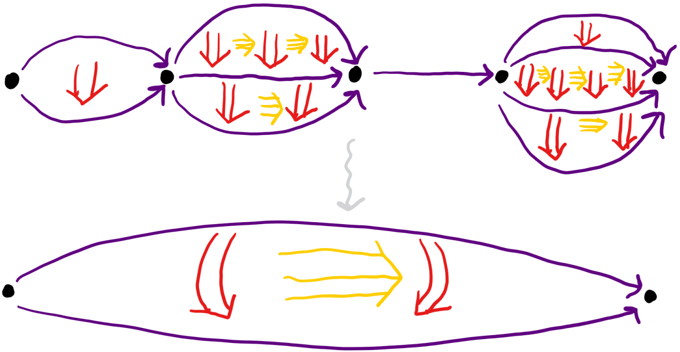

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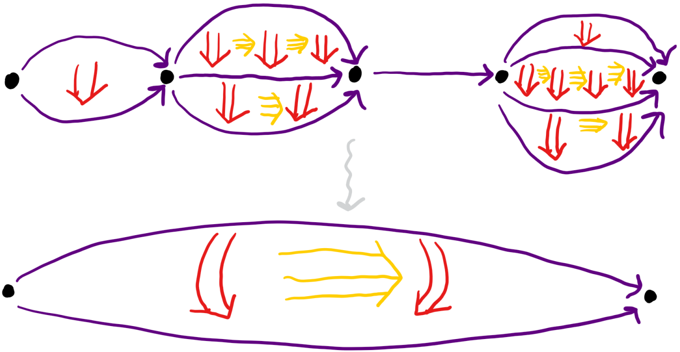

Strict ω-Categories

- For each pasting diagram shape D,

$$Hom(D, A) := \{\text{diagrams of shape } D \text{ in } A\}$$

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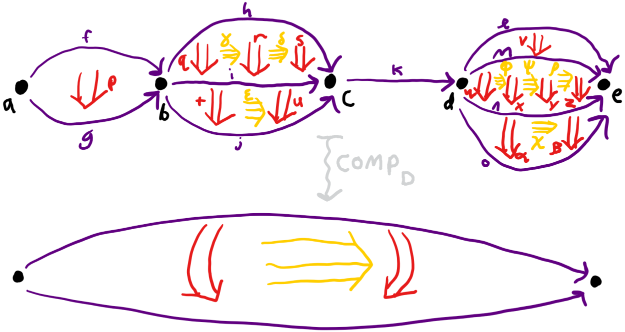
...with composition maps for all D:

$$comp_D : Hom(D, A) \to A_n$$



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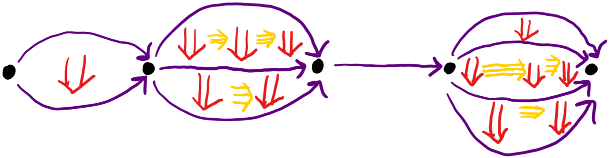

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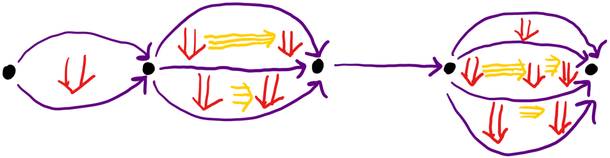

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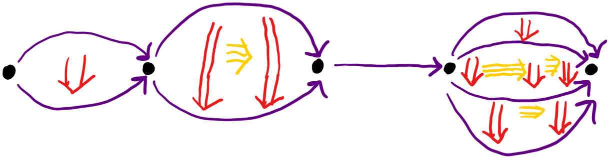

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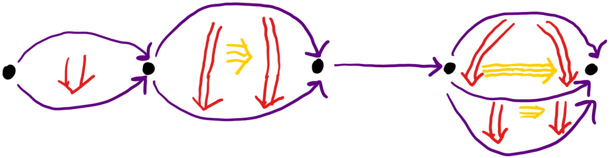

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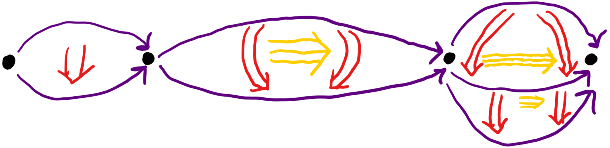

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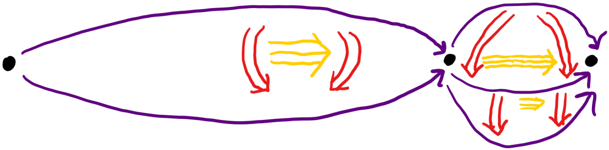

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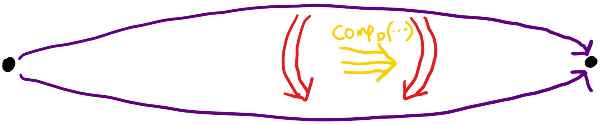


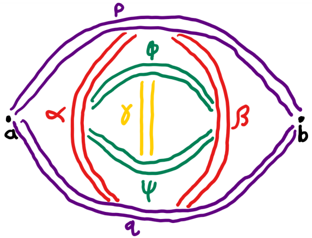
All composition orders give the same result

Types

- A type A forms a globular set:

$$A \underset{t}{\overset{s}{\leftleftarrows}} \sum_{a,b:A} a = b \underset{t}{\overset{s}{\leftleftarrows}} \sum_{a,b:A} \sum_{p,q:a=b} p = q \underset{t}{\overset{s}{\leftleftarrows}} \cdots$$

- A will have compositions, but not strict associativity



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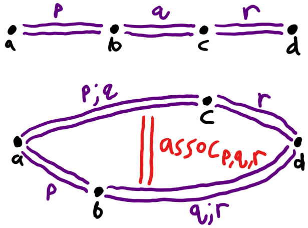
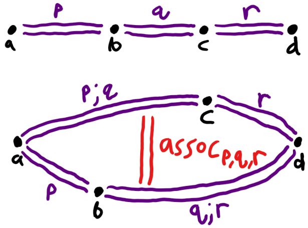
$$A \underset{t}{\overset{s}{\leftleftarrows}} \sum_{a,b:A} a = b \underset{t}{\overset{s}{\leftleftarrows}} \sum_{a,b:A} \sum_{p,q:a=b} p = q \underset{t}{\overset{s}{\leftleftarrows}} \cdots$$

- A will have compositions, but not strict associativity
- Different composition orders for diagrams of n-cells are not the same
- But they are related by $(n+1)$-cells



Globular Operads

- How can we describe this weak associativity?
- What are "composition orders" ?



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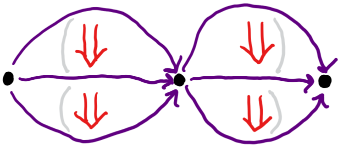
- A globular operad is a set P_D of "evaluation strategies" for each pasting diagram of shape D
- These strategies must allow "substitution"



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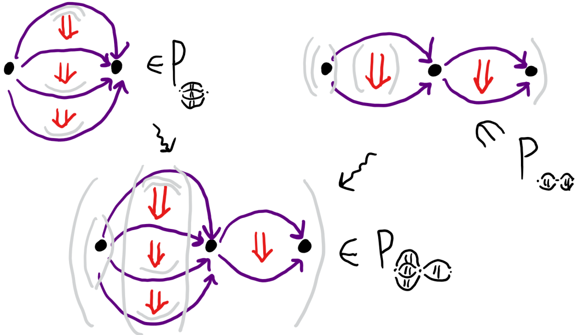
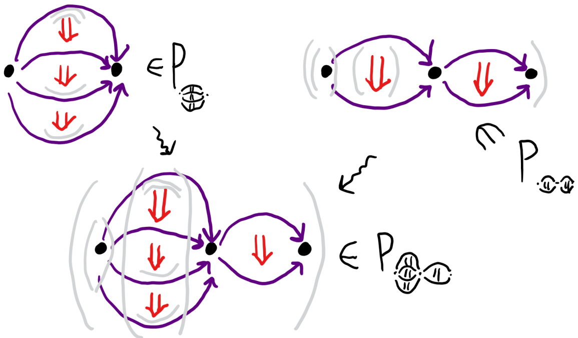


Globular Operads

- Each evaluation strategy in P_D gives a different composition
- An ω_P-category (or P-algebra) is a globular set A with

$$comp_D : P_D \to Hom(D, A) \to A_n$$

for each free pasting shape D, compatible with substitution



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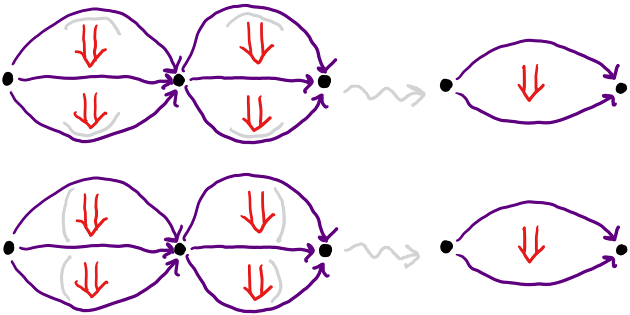

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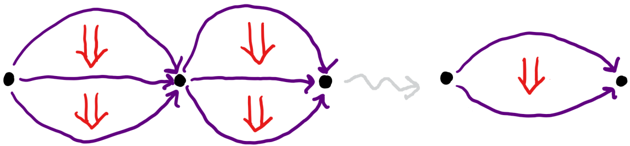
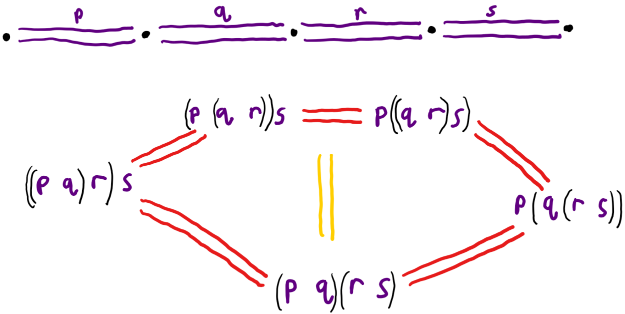
- Then an ω_P-category is just a strict ω-category



Contractibility

- An ω_P-category A has $comp_D : P_D \to Hom(D, A) \to A_n$
- If all $P_D = *$, then an ω_P-category is just a strict ω-category
- Types have all properties of strict ω-categories up to higher cells



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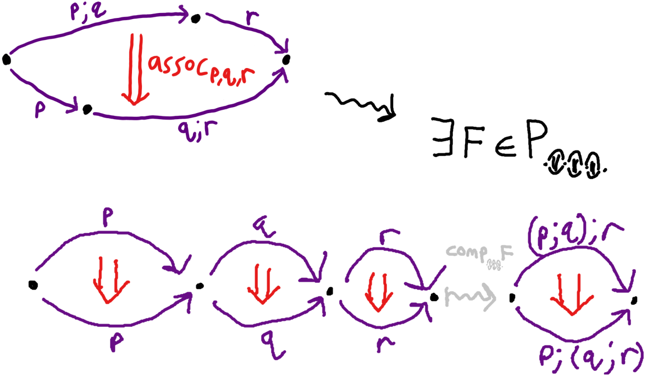

Weak ω-Categories

- Types have all properties of strict ω-categories up to higher cells
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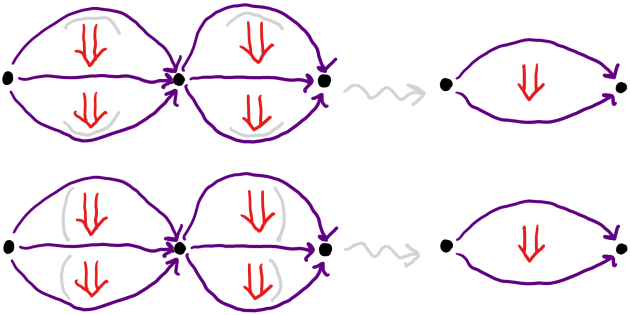
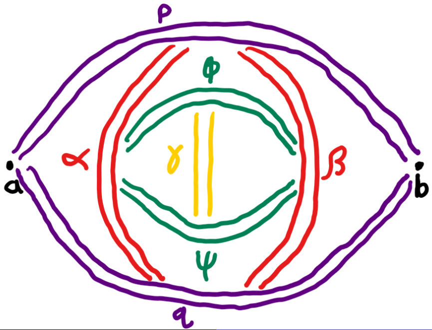


Types are Weak ω-Categories

- A type A and its nested equalities form a globular set
- Path induction gives us all conceivable compositions built from reflexivity on identity paths



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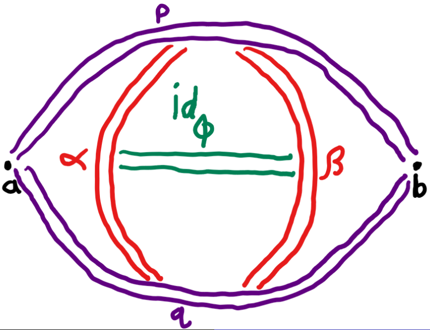

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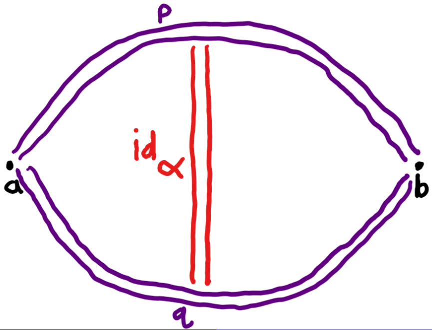

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Types are Weak ω-Categories

- A type A and its nested equalities form a globular set
- Path induction gives us all conceivable compositions built from reflexivity on identity paths
- These compositions form a normalized globular operad P_A
- Path induction lets us show P_A is contractible
- A thus forms a weak ω-category
- Types also have weak inverses, so A is a weak ω-groupoid
- Is that all???

Fin

Thank you!

References

- Weak ω-groupoids in type theory:
  Benno van den Berg, Richard Garner. Types are Weak ω-Groupoids.
- More definitions of higher categories:
  Tom Leinster. Higher Operads, Higher Categories.