Cubes as Test Categories

Tuesday, May 5, 2020 3:27 AM

— Classically, simplicial / cubical sets model spaces / ∞-groupoids

— simplicial sets more convenient

- Products respect homotopy types
- Nerve functor from categories

— How "well" do $\hat{\Delta}$, $\hat{\square}$, $\hat{\Theta}$, $\hat{\Omega}$, $\hat{\Delta}_s$, & other cubical sets model spaces?

— A is a (weak / local / - / strict) test category when $\hat{A}$ models the homotopy theory of spaces with certain properties

Homotopy Theory of Categories

— A functor $F : \mathcal{C} \to \mathcal{D}$ is a weak equivalence of categories if

$NF : N\mathcal{C} \to N\mathcal{D}$ is a weak equivalence of simplicial sets

— A natural transformation $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd}$ induces a homotopy $N\mathcal{C} \times \Delta^1 \to N\mathcal{D}$ from F to G

— Any functor $F : \mathcal{C} \to \mathcal{C}$ with a zigzag $F \Rightarrow \cdots \Leftarrow id$ to the identity is a w.e.

— $\mathcal{C}$ is contractible if $\mathcal{C} \to *$ is a w.e. (eg if $\mathcal{C}$ has a terminal object)

— (Quillen Theorem A) For $F : \mathcal{C} \to \mathcal{D}$ if all F/d are contractible ($d \in ob\,\mathcal{D}$) F is a w.e.

— (Thomason) The homotopy theory of categories is equivalent to spaces

Homotopy in $\hat{A}$

— Consider $i_A : A \to Cat : a \mapsto A/a$

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[dr, "y"'] & Cat \\ & \hat{A} \arrow[u, "el"'] \arrow[ul, phantom, "\cong"] \end{tikzcd}$$

is a left Kan extension

$el : \hat{A} \to Cat$ sends X to its category of elements

Ex If $A = \square_0$ and $X = \square^2$, $el(X) =$

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & \bullet \arrow[l] \arrow[d] \\ \bullet \arrow[r] & \bullet & \bullet \arrow[l] \\ \bullet \arrow[r] \arrow[u] & \bullet \arrow[u] & \bullet \arrow[l] \arrow[u] \end{tikzcd} \cong \square_0/\square^2$$

— $f : X \to Y$ is a weak equivalence in $\hat{A}$ if $el(f) : el(X) \to el(Y)$ is a w.e. in Cat

X is contractible if $el(X)$ is contractible

<u>Ex</u> Each representable a in $\hat{A}$ is contractible as $el(a) \cong A/a$, which has a terminal object

<u>Ex</u> For $*$ terminal in $\hat{A}$, $el(*) \cong A$, so $*$ is contractible in $\hat{A}$ iff A is contractible in Cat

— i_A also determines a functor $i_A^* : Cat \to \hat{A} : C \mapsto (a \mapsto Fun(A/a, C))$

Test Categories

— A is a weak test category if $el(i_A^* C) \to C$ is a w.e. for all C
($\hat{A}$ and Cat have equivalent homotopy categories Hot)

— A is a test category if A is contractible and each A/a is a weak test cat

— A is a strict test category if furthermore $\hat{A} \to$ Hot preserves products

<u>Ex</u> Δ is a strict test category
Δ_s is a weak test category

— A separated interval in $\hat{A}$ is a pullback square

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & I \end{tikzcd}$$

<u>Ex</u>

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \square^0 \cong * \arrow[d, "\partial^1_{1,1}"] \\ \square^0 \arrow[r, "\partial^1_{1,0}"'] & \square^1 \end{tikzcd}$$

is a separated interval in $\hat{\square}_a$ if $\partial, \sigma \in a$ and $\rho \notin a$

— (Grothendieck) A is a strict test category iff

- $\hat{A}$ has a separated contractible interval
- $a \times b$ is contractible in $\hat{A}$ for all a, b in A

Some Cube categories are strict test

— Any Cartesian cube category $(\square_a, w, \tau, \sigma \in a)$ is a strict test category

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- Has a separated contractible interval
- $a \times b$ is representable, hence contractible

— (Maltsiniotis) Any cube category with degeneracies and either type of connections ($\sigma, \gamma_{0 \text{ or } 1} \in a$) is a strict test category

Proof
- Has a separated contractible interval
- will show $\square^m \times \square^n \xrightarrow{id \times \sigma_1} \square^m \times \square^{n-1} \xrightarrow{id \times \sigma_1} \dots \xrightarrow{id \times \sigma_1} \square^m \times \square^0 \cong \square^m$ is a w.e.

  by showing $el(\square^m \times \square^{n+1}) \xrightarrow{el(id \times \sigma_1)} el(\square^m \times \square^n)$ is a homotopy equivalence in Cat

  — homotopy inverse $\xleftarrow[el(id \times \partial_{1,1})]{}$

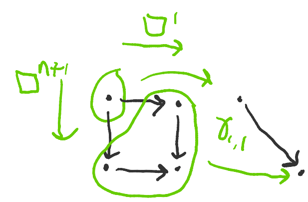
  — compose to identity on $el(\square^m \times \square^n)$

  — suffices to find homotopy from id to $el(id \times \partial_{1,1}\sigma_1)$ in $el(\square^m \times \square^{n+1})$

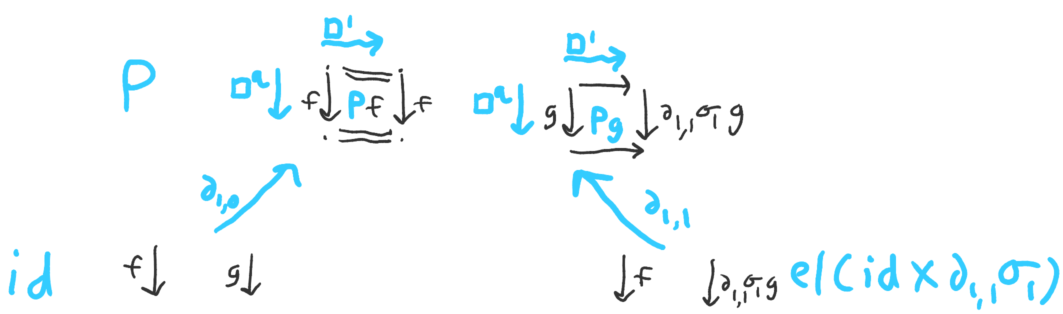
$el(\square^m \times \square^{n+1})$ has objects $(\square^a \xrightarrow{f} \square^m, \square^a \xrightarrow{g} \square^{n+1})$

For any f, g, define $Pf : \square^{a+1} \xrightarrow{\sigma_1} \square^a \xrightarrow{f} \square^m$

$Pg : \square^{a+1} \xrightarrow{id_{\square^1} \otimes g} \square^{n+2} \xrightarrow{\gamma_{1,1}} \square^{n+1}$



$P : el(\square^m \times \square^{n+1}) \longrightarrow el(\square^m \times \square^{n+1})$



$id \Rightarrow P \Leftarrow el(id \times \partial_{1,1}\sigma_1)$ □

The remaining cube categories

— $i_A^* : Cat \to \hat{A}$ is not the only nerve functor

Any $i : A \to Cat$ defines $i^* : Cat \to \hat{A} : c \mapsto (a \mapsto Fun(i(a), c))$

— Let $2 = \bullet \longrightarrow \bullet$ in Cat

— (Grothendieck) If A is contractible and $i : A \to \mathrm{Cat}$ is s.t.

- $i(a)$ has a terminal object for all a
- $\mathrm{el}(a \times i^*(2))$ is contractible

then A is a test category

— (Cisinski) All cube categories are test categories

<u>Proof</u>
- $\square_a$ is contractible when $\sigma \in a$
- Let $i(\square^n) = 2^n$ (or $(\cdot \rightleftarrows \cdot)^n$ if there are reversals)
- 2^n has a terminal object
- will show $\mathrm{el}(\square^n \times i^*(2)) \xrightarrow{pr} \mathrm{el}(\square^n) \simeq \square_a/_{\square^n}$ is a homotopy equivalence

$\mathrm{el}(\square^n \times i^*(2))$ has objects $\left(\square^a \xrightarrow{f} \square^n,\ 2^q \xrightarrow{g} 2\right)$

homotopy inverse $\square_a/_{\square^n} \xrightarrow{const} \mathrm{el}(\square^n \times i^*(2))$

$\square^a \xrightarrow{f} \square^n \mapsto (f, const_1 : 2^q \to 2)$

$pr \cdot const = id$, want homotopy from id to $const \cdot pr$

Define $P(f,g) = \left(\square^{q+1} \xrightarrow{\sigma_1} \square^q \xrightarrow{f} \square^n,\ 2^{q+1} \cong 2 \times 2^q \longrightarrow 2\right)$

$(0, x) \longmapsto g(x)$

$(1, x) \longmapsto 1$

P

$$f \downarrow \overline{\overline{\underline{\underline{Pf}}}} \downarrow f \qquad 2^q \downarrow g \downarrow \underset{\longrightarrow}{\overset{\overset{2}{\longrightarrow}}{\longrightarrow}} Pg \downarrow const_1$$

id $\quad f, g \qquad\qquad f, const_1 \quad$ $const\ pr$

$id \Rightarrow P \Leftarrow const \cdot pr$ $\square$

— (Maltsiniotis, Buchholtz–Morehouse)

$\square_{\partial\sigma}$, $\square_{\partial\sigma\gamma}$, $\square_{\partial\sigma\rho}$, $\square_{\partial\sigma\gamma\rho}$ are not strict test categories

<u>Proof</u> In the case of $\square_{\partial\sigma}$, $\square^1$ contains $0 \xrightarrow{x} 1$

$\mathrm{el}(\square^1 \times \square^1)$ is (by Theorem A) weakly equivalent to the full subcategory on the elements

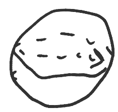
Full subcategory on the elements

$$\{(0,0),\ (0,1),\ (1,0),\ (1,1)\} = (\square^1 \times \square^1)_0$$

$$\{(\sigma_1 0, x),\ (\sigma_1 1, x),\ (x, \sigma_1 0),\ (x, \sigma_1 1),\ (x, x)\} = (\square^1 \times \square^1)_1$$

$$\{\underbrace{(\sigma_1 x, \sigma_2 x)}_{y_1},\ \underbrace{(\sigma_2 x, \sigma_1 x)}_{y_2}\} \subseteq (\square^1 \times \square^1)_2$$

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] \arrow[dr] & (0,1) \arrow[d] \\
(1,0) \arrow[r] & (1,1)
\end{tikzcd}
\quad \text{with} \quad
\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] \arrow[dr, phantom, "y_1"] & (0,1) \arrow[d] \\
(1,0) \arrow[r] & (1,1)
\end{tikzcd}
\quad
\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] \arrow[dr, phantom, "y_2"] & (0,1) \arrow[d] \\
(1,0) \arrow[r] & (1,1)
\end{tikzcd}$$

$\simeq$  $\simeq S^2 \vee S^1$