Comparing Shapes for Higher Structures

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Cornell University

Young Topologists Meeting 2018

Outline

- What are cell structures?
- How can they describe higher categories?

Simplicial Sets

The simplex category Δ is the category of finite nonempty ordinals and order preserving functions:

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$$\Delta := \{0\} \; \begin{matrix} \xrightarrow{d^0} \\ \xleftarrow{s^0} \\ \xrightarrow{d^1} \end{matrix} \; \{0,1\} \; \begin{matrix} \xrightarrow{d^0} \\ \xleftarrow{s^0} \\ \xrightarrow{d^1} \\ \xleftarrow{s^1} \\ \xrightarrow{d^2} \end{matrix} \; \{0,1,2\} \qquad \cdots$$

d^i skips i, s^i repeats i, and these generate all maps in Δ

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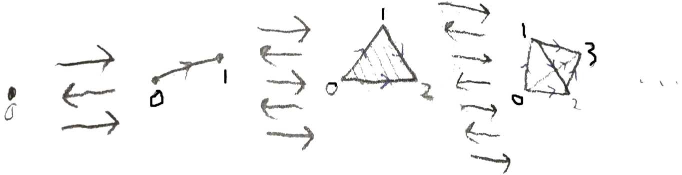
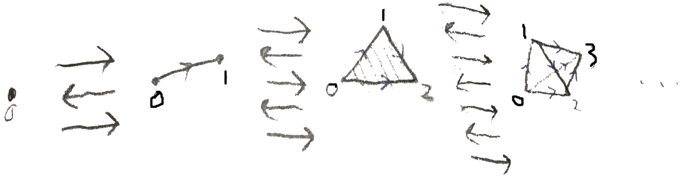
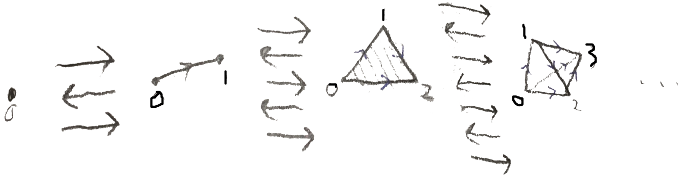


where the maps act on the vertices as in Δ and extend linearly

Simplicial Sets



Simplicial Sets

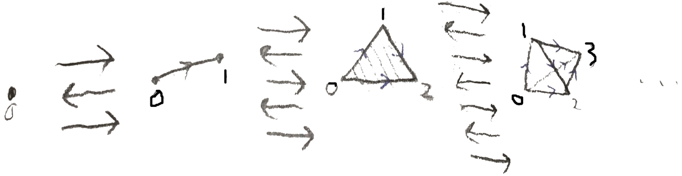


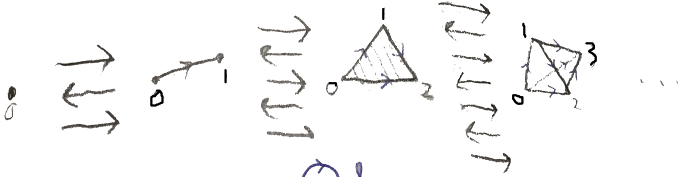
Idea:

$$
\begin{tikzcd}[column sep=huge]
X = & X_0 \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] \arrow[d, equal] & X_1 \arrow[r, leftarrow, shift left=4] \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[r, leftarrow, shift right=4] \arrow[d, equal] & X_2 \arrow[r, leftarrow, shift left=6] \arrow[r, shift left=4] \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] \arrow[r, shift right=4] \arrow[r, leftarrow, shift right=6] \arrow[d, equal] & X_3 \arrow[d, equal] & \cdots \\
& \{0\text{-simplices}\} & \{1\text{-simplices}\} & \{2\text{-simplices}\} & \{3\text{-simplices}\} & \cdots
\end{tikzcd}
$$

'face maps' d_i, 'degeneracy maps' s_i

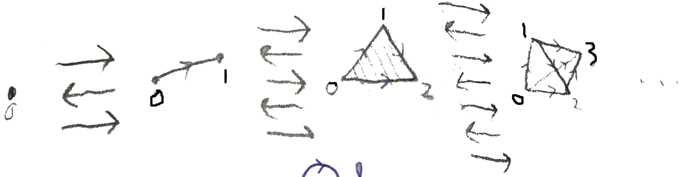
Simplicial Sets





Example: Simplicial Circle

$$X = \begin{array}{ccccccc} X_0 & \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} & X_1 & \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} & X_2 & & \cdots \\ \| & & \| & & \| & & \\ \{b\} & & \{\ell, s_0 b\} & & \{sb, s_0\ell, s_1\ell\} & & \cdots \end{array}$$



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Cubical Sets

The cube category $\square$ is the category:

$$\square := 0 \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} 1 \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} 2 \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} 3 \qquad\qquad \cdots$$

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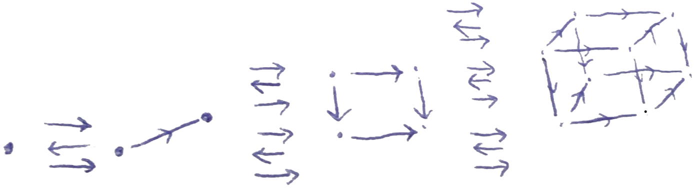
with composition described via a functor $\square \to \mathit{Top}$:



where the maps are two face inclusions and one projection in each dimension

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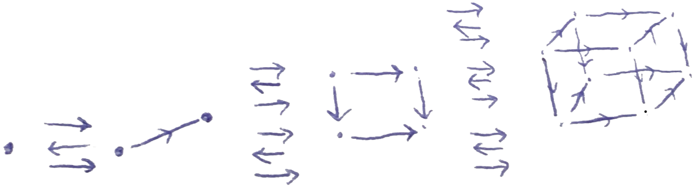

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A cubical set is a functor $\square^{op} \to \mathit{Set}$

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The glob(e) category G is the category:

$$G := 0 \underset{\bar{t}}{\overset{\bar{s}}{\rightrightarrows}} 1 \underset{\bar{t}}{\overset{\bar{s}}{\rightrightarrows}} 2 \underset{\bar{t}}{\overset{\bar{s}}{\rightrightarrows}} 3 \qquad \cdots$$

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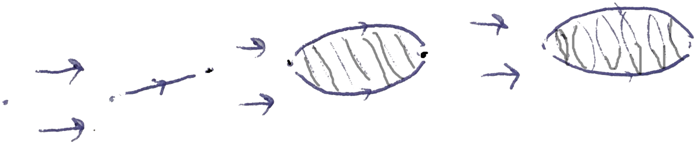
G can be realized in *Top* by lemons:



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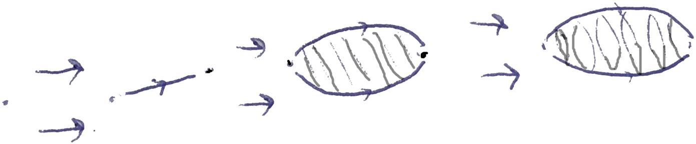

but these don't capture the directionality

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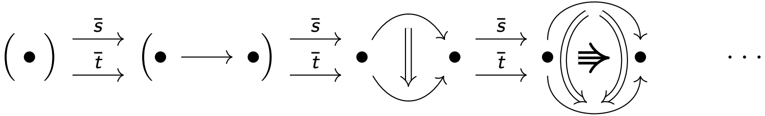

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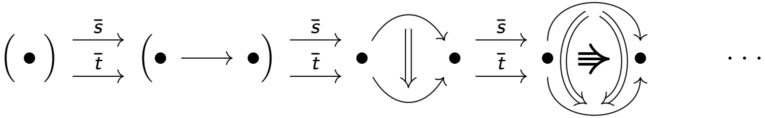

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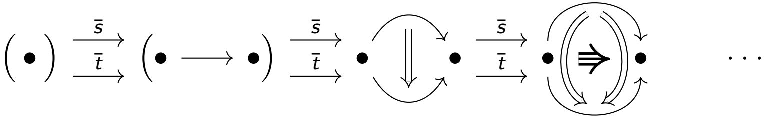

A globular set X is a functor $G^{op} \to Set$

$$X = X_0 \underset{t}{\overset{s}{\leftleftarrows}} X_1 \underset{t}{\overset{s}{\leftleftarrows}} X_2 \underset{t}{\overset{s}{\leftleftarrows}} X_3 \qquad \cdots$$

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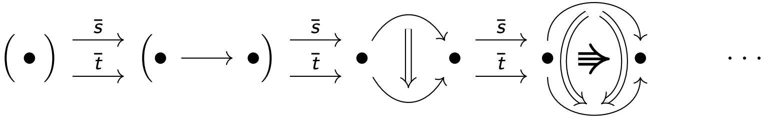

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"a collection of things in each dimension having source and target with fixed boundary"

Algebraic Composition

- Where are the categories?

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- How does this work with higher dimensions?

Algebraic Composition

Dimension 1:

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$$X = X_0 \underset{t}{\overset{s}{\leftleftarrows}}\!\!\xrightarrow{i} X_1 \underset{t}{\overset{s}{\leftleftarrows}}\!\!\xrightarrow{i} X_2 \underset{t}{\overset{s}{\leftleftarrows}}\!\!\xrightarrow{i} X_3 \qquad \cdots$$

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strict

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, "(f;g);h"] \arrow[r, bend right=60, "f;(g;h)"'] \arrow[r, phantom, "\|"] & \bullet \end{tikzcd}$$

weak

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, "(f;g);h"] \arrow[r, bend right=60, "f;(g;h)"'] \arrow[r, phantom, "\Downarrow\ \Uparrow"] & \bullet \end{tikzcd}$$

weaker

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, "(f;g);h"] \arrow[r, bend right=60, "f;(g;h)"'] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

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Similar choices for unit laws

Algebraic Composition

Dimension 2:

Algebraic Composition

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$$
\begin{tikzcd}
\bullet \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \bullet
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\bullet \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, "\Downarrow\alpha;\beta"] & \bullet
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$$

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\begin{tikzcd}
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\end{tikzcd}
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\begin{tikzcd}
\bullet \arrow[r, bend left=60, "f;g"] \arrow[r, bend right=60, "h;k"'] \arrow[r, phantom, "\Downarrow\alpha\cdot\beta"] & \bullet
\end{tikzcd}
$$

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$$
\begin{tikzcd}
\bullet \arrow[r, bend left=60, "f"] \arrow[r, "g"] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3ex, "\Downarrow\beta"] & \bullet
\end{tikzcd}
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\end{tikzcd}
$$

Like a 2-category!

Algebraic Composition

Dimension 2:

$$\begin{tikzcd}[column sep=huge] \bullet \arrow[r, bend left=60, "f"{name=F}] \arrow[r, "g"{name=G, description}] \arrow[r, bend right=60, "h"'{name=H}] & \bullet \arrow[Rightarrow, from=F, to=G, "\alpha"] \arrow[Rightarrow, from=G, to=H, "\beta"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd}[column sep=huge] \bullet \arrow[r, bend left=60, "f"{name=F}] \arrow[r, bend right=60, "h"'{name=H}] & \bullet \arrow[Rightarrow, from=F, to=H, "\alpha;\beta"] \end{tikzcd}$$

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Like a 2-category!

Associativity, identity conditions can similarly be strict or (various forms of) weak

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\begin{tikzcd}
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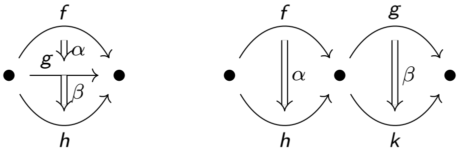
Higher dimensions even more complicated, especially weak versions

Algebraic Composition



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This is all algebraic structure on an underlying globular set

Algebraic Composition

$$
\begin{tikzcd}
\bullet \arrow[r, bend left=60, "f"{name=F}] \arrow[r, "g"{name=G, description}] \arrow[r, bend right=60, "h"'{name=H}] & \bullet
\arrow[Rightarrow, from=F, to=G, "\alpha"]
\arrow[Rightarrow, from=G, to=H, "\beta"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, bend left=60, "f"{name=F}] \arrow[r, bend right=60, "h"'{name=H}] & \bullet \arrow[r, bend left=60, "g"{name=G}] \arrow[r, bend right=60, "k"'{name=K}] & \bullet
\arrow[Rightarrow, from=F, to=H, "\alpha"]
\arrow[Rightarrow, from=G, to=K, "\beta"]
\end{tikzcd}
$$

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A globular set with (some sort of) algebraic composition structure is called (some version of) an ∞-category

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$$
\begin{tikzcd}
\bullet \arrow[r, bend left=60, "f"] \arrow[r, "g"] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=2ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2ex, "\Downarrow\beta"] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, "\Downarrow\alpha"] & \bullet \arrow[r, bend left=60, "g"] \arrow[r, bend right=60, "k"'] \arrow[r, phantom, "\Downarrow\beta"] & \bullet
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(Often called an (∞, ∞)-category)

Algebraic Composition

$$\begin{tikzcd}[column sep=large] \bullet \arrow[r, bend left=60, "f", ""{name=F, below}] \arrow[r, "g"{near start}, ""{name=G}] \arrow[r, bend right=60, "h"', ""{name=H, above}] & \bullet \arrow[Rightarrow, from=F, to=G, "\alpha"] \arrow[Rightarrow, from=G, to=H, "\beta"] \end{tikzcd} \qquad \begin{tikzcd}[column sep=large] \bullet \arrow[r, bend left=60, "f", ""{name=F, below}] \arrow[r, bend right=60, "h"', ""{name=H, above}] & \bullet \arrow[r, bend left=60, "g", ""{name=G, below}] \arrow[r, bend right=60, "k"', ""{name=K, above}] & \bullet \arrow[Rightarrow, from=F, to=H, "\alpha"] \arrow[Rightarrow, from=G, to=K, "\beta"] \end{tikzcd}$$

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An (∞, n)-category is an ∞-category in which all cells of dimension $> n$ have (some kind of) inverses

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Example: *Top*

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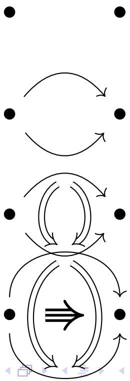
Example: *Top*

$$Top_0 = \text{(some nice set of) } \textit{spaces}$$

$$Top_1 = \textit{continuous functions}$$

$$Top_2 = \textit{homotopies}$$

$$Top_3 = \textit{homotopies of homotopies}$$



Geometric Composition

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What about simplicial sets?

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plus similar 'filling' conditions in higher dimensions to give associativity

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Simplicial sets can only model $(\infty, 1)$ in this way

Thanks!

The End