Comparing Shapes for Higher Structures

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Outline

- What are cell structures?
- How can they describe higher categories?

Simplicial Sets

The simplex category Δ is the category of finite nonempty ordinals and order preserving functions:

$$\Delta := \{0\} \underset{\xrightarrow{d^1}}{\overset{\xrightarrow{d^0}}{\xleftarrow{s^0}}} \{0,1\} \quad \begin{matrix} \xrightarrow{d^0} \\ \xleftarrow{s^0} \\ \xrightarrow{d^1} \\ \xleftarrow{s^1} \\ \xrightarrow{d^2} \end{matrix} \quad \{0,1,2\} \qquad \cdots$$

d^i skips i, s^i repeats i, and these generate all maps in Δ

A simplicial set X is a functor $\Delta^{op} \to Set$:

$$X = X_0 \underset{\xleftarrow{d_1}}{\overset{\xleftarrow{d_0}}{\xrightarrow{s_0}}} X_1 \quad \begin{matrix} \xleftarrow{d_0} \\ \xrightarrow{s_0} \\ \xleftarrow{d_1} \\ \xrightarrow{s_1} \\ \xleftarrow{d_2} \end{matrix} \quad X_2 \qquad \cdots$$

Simplicial Sets

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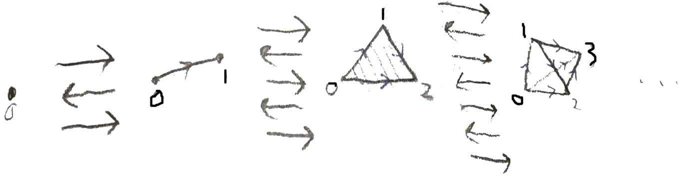
$$\Delta := \{0\} \begin{array}{c} \xrightarrow{d^0} \\ \xleftarrow{s^0} \\ \xrightarrow{d^1} \end{array} \{0,1\} \begin{array}{c} \xrightarrow{d^0} \\ \xleftarrow{s^0} \\ \xrightarrow{d^1} \\ \xleftarrow{s^1} \\ \xrightarrow{d^2} \end{array} \{0,1,2\} \qquad \cdots$$

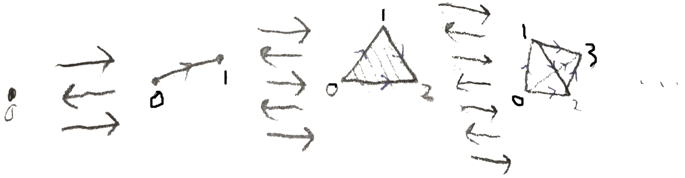
d^i skips i, s^i repeats i, and these generate all maps in Δ

There is a functor $\Delta \to \mathit{Top}$:



where the maps act on the vertices as in Δ and extend linearly

Simplicial Sets



Idea:

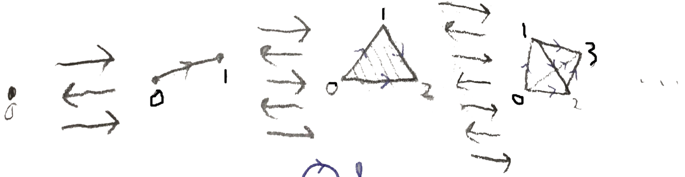
$$X = \quad X_0 \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; X_1 \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; X_2 \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; X_3 \quad \cdots$$

$$\| \qquad \| \qquad \| \qquad \|$$

$$\{\text{0-simplices}\} \quad \{\text{1-simplices}\} \quad \{\text{2-simplices}\} \quad \{\text{3-simplices}\} \quad \cdots$$

'face maps' d_i, 'degeneracy maps' s_i

Simplicial Sets



Example: Simplicial Circle

$$X = \quad X_0 \;\leftleftarrows\!\!\rightarrow\; X_1 \;\;\leftleftarrows\!\!\rightrightarrows\!\!\leftarrow\;\; X_2 \qquad \cdots$$

$$\| \qquad\qquad \| \qquad\qquad \|$$

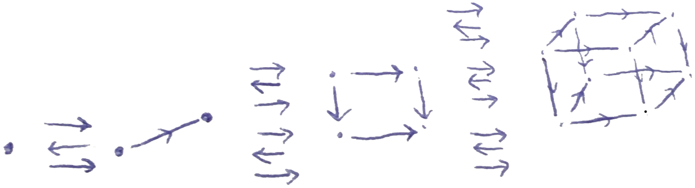
$$\{b\} \qquad \{\ell, s_0 b\} \qquad \{sb, s_0\ell, s_1\ell\} \qquad \cdots$$

Cubical Sets

The cube category $\square$ is the category:

$$\square := 0 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; 1 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; 2 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; 3 \qquad \cdots$$

with composition described via a functor $\square \to Top$:



where the maps are two face inclusions and one projection in each dimension

A cubical set is a functor $\square^{op} \to Set$

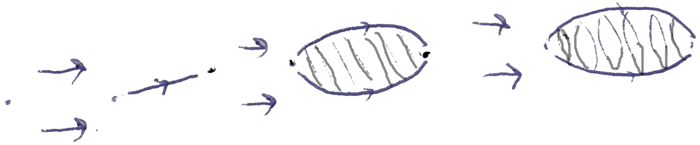
Globular Sets

The glob(e) category G is the category:

$$G := 0 \underset{\bar{t}}{\overset{\bar{s}}{\rightrightarrows}} 1 \underset{\bar{t}}{\overset{\bar{s}}{\rightrightarrows}} 2 \underset{\bar{t}}{\overset{\bar{s}}{\rightrightarrows}} 3 \qquad \cdots$$

with $\bar{s} \circ \bar{s} = \bar{t} \circ \bar{s}$ and $\bar{s} \circ \bar{t} = \bar{t} \circ \bar{t}$

G can be realized in *Top* by lemons:



but these don't capture the directionality

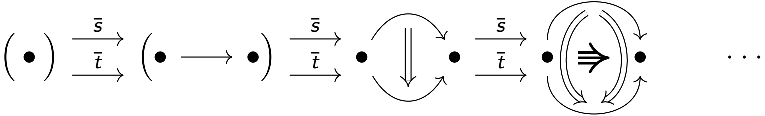
Globular Sets

The glob(e) category G is the category:

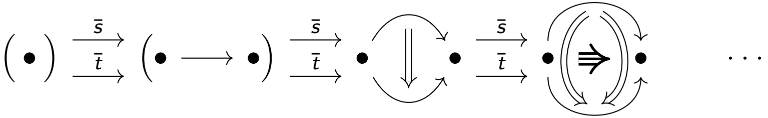
$$G := 0 \underset{\bar{t}}{\overset{\bar{s}}{\rightrightarrows}} 1 \underset{\bar{t}}{\overset{\bar{s}}{\rightrightarrows}} 2 \underset{\bar{t}}{\overset{\bar{s}}{\rightrightarrows}} 3 \qquad \cdots$$

with $\bar{s} \circ \bar{s} = \bar{t} \circ \bar{s}$ and $\bar{s} \circ \bar{t} = \bar{t} \circ \bar{t}$

Instead, think of globular cells as arrows between arrows (glob(e)s):



Globular Sets



A globular set X is a functor $G^{op} \to Set$

$$X = X_0 \underset{t}{\overset{s}{\leftleftarrows}} X_1 \underset{t}{\overset{s}{\leftleftarrows}} X_2 \underset{t}{\overset{s}{\leftleftarrows}} X_3 \qquad \cdots$$

where $s \circ s = s \circ t$ and $t \circ s = t \circ t$

"a collection of things in each dimension having source and target with fixed boundary"

Algebraic Composition

- Where are the categories?
- Things with source and target are ripe for composition!
- How does this work with higher dimensions?

Algebraic Composition

Dimension 1:

$$x \xrightarrow{f} y \xrightarrow{g} z \rightsquigarrow x \xrightarrow{f;g} z$$

Like a category!

Identities added either as algebraic structure or to cell structure:

$$G' := 0 \; \overset{\bar{s}}{\longrightarrow} \; \overset{\bar{i}}{\longleftarrow} \; \overset{\bar{t}}{\longrightarrow} \; 1 \; \overset{\bar{s}}{\longrightarrow} \; \overset{\bar{i}}{\longleftarrow} \; \overset{\bar{t}}{\longrightarrow} \; 2 \; \overset{\bar{s}}{\longrightarrow} \; \overset{\bar{i}}{\longleftarrow} \; \overset{\bar{t}}{\longrightarrow} \; 3 \qquad \cdots$$

$$X = X_0 \; \overset{s}{\longleftarrow} \; \overset{i}{\longrightarrow} \; \overset{t}{\longleftarrow} \; X_1 \; \overset{s}{\longleftarrow} \; \overset{i}{\longrightarrow} \; \overset{t}{\longleftarrow} \; X_2 \; \overset{s}{\longleftarrow} \; \overset{i}{\longrightarrow} \; \overset{t}{\longleftarrow} \; X_3 \qquad \cdots$$

Algebraic Composition

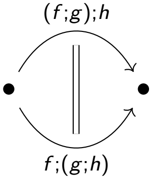
Dimension 1:

$$x \xrightarrow{f} y \xrightarrow{g} z \rightsquigarrow x \xrightarrow{f;g} z$$

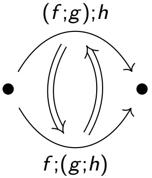
Like a category!

Associativity can come in many forms:

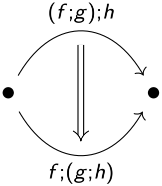
strict



weak



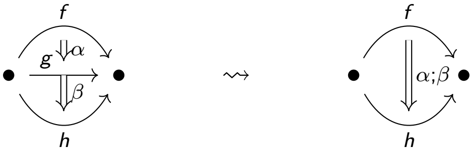
weaker

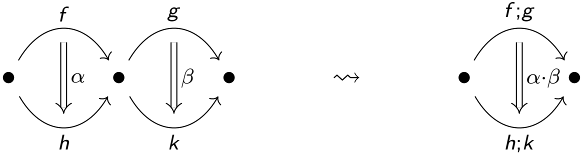


Similar choices for unit laws

Algebraic Composition

Dimension 2:

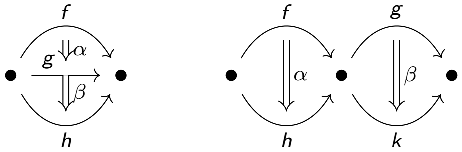




Like a 2-category!

Associativity, identity conditions can similarly be strict or (various forms of) weak

Algebraic Composition



Higher dimensions even more complicated, especially weak versions

This is all algebraic structure on an underlying globular set

A globular set with (some sort of) algebraic composition structure is called (some version of) an ∞-category

(Often called an (∞, ∞)-category)

An (∞, n)-category is an ∞-category in which all cells of dimension $> n$ have (some kind of) inverses

Algebraic Composition

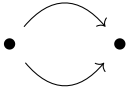
A globular set with (some sort of) algebraic composition structure is called (some version of) an ∞-category

An (∞, n)-category is an ∞-category in which all cells of dimension $> n$ have (some kind of) inverses

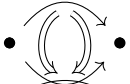
Example: *Top*

$Top_0 =$ (some nice set of) *spaces*

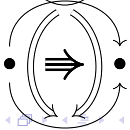
$Top_1 =$ *continuous functions*



$Top_2 =$ *homotopies*



$Top_3 =$ *homotopies of homotopies*



Geometric Composition

Algebraic composition structure is tough to work with

What about simplicial sets?

plus similar 'filling' conditions in higher dimensions to give associativity

These 'quasicategories' are 'equivalent' to $(\infty, 1)$-categories

Simplicial sets can only model $(\infty, 1)$ in this way

Thanks!

The End