A Pointless Topological Equivalence

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1. Introduction

Two topological spaces are most commonly considered to be equivalent if there is a homeomorphism between them; a function from the points of one space to those of another that happens to preserve the topological structure of the spaces. However, a topological space is characterized by both a set of points and a collection of subsets of those points which are closed under unions and finite intersections. This collection of "open" subsets can be described by a special kind of lattice, which can be used to define a different equivalence relation of topological spaces; namely, two spaces are equivalent when they have isomorphic lattices of open sets. It is then natural to consider how these two notions of equivalence relate to each other. For instance, what kinds of relationships exist between spaces with isomorphic open subset lattices? Are they always homeomorphic? These relationships have been studied by category theorists (for instance, Stone Duality), relating the study of topology with points in a space as the fundamental objects to that with open sets as fundamental outside of the context of the points they contain (sometimes called "Pointless Topology")(3). These topics rely heavily on category and lattice theory, but here I will describe a similar equivalence of topological spaces using only categories and functors, and show that for a significant class of spaces it is equivalent to homeomorphism.

2. Categorically Isomorphic Spaces

A topology on a set X is a subcollection of the power set of X, the elements of which are called open sets. It is straightforward to show that this collection of open sets forms a category, where there is a morphism from each open set to each open set containing it (1).

Definition 1. For topological space X, let Op_X denote the category of open sets of X, where for U, V open in X, $Hom(U, V)$ is the singleton set if $U \subseteq V$ and empty otherwise.

From this definition comes the desired equivalence of topological spaces, as two spaces can be compared by their categories of open sets. There is more than one standard equivalence relation on categories, but here isomorphism is the most intuitive to consider (4).

Definition 2. Categories $\mathcal{C}$ and $\mathcal{D}$ are isomorphic if there exist functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G$ is the identity functor on $\mathcal{C}$ and $G \circ F$ is the identity functor on $\mathcal{D}$. G can then be written as F^{-1}.

Definition 3. Topological spaces X and Y are categorically isomorphic if there is an isomorphism of categories between Op_X and Op_Y. Unpacking this statement, X and Y are categorically isomorphic if there is a bijective function F from the open sets of X to those of Y such that if $U \subseteq V$ in X, $F(U) \subseteq F(V)$ in Y. The inverse $F^{-1} : Op_Y \to Op_X$ satisfies the same property.

As morphisms between pairs of open sets are unique (Hom is always a singleton set if nonempty), the functor laws reduce to guaranteeing the existence of a morphism between open sets in Y if there is a morphism between their preimages in X (composition is satisfied trivially). This simplification of the functor laws suggests that giving the structure of a category to a topology is redundant, as a simpler algebraic structure would suffice (3).

3. Functor Induced by Homeomorphism

Proposition 1. *Homeomorphic topological spaces are categorically isomorphic.*

Proof Let X and Y be topological spaces and $f : X \to Y$ be a homeomorphism. Define $F : Op_X \to Op_Y$ such that for open $U \subseteq X$, $F(U) = f(U)$, which is open in Y as f is a an open map. As f is injective, if U and V are distinct subsets of X, then $f(U)$ and $f(V)$ are also distinct, so F is injective. As f is continuous, every open set in Y is the image of an open set in X, so F is surjective. If $U \subseteq V$ in X, then $f(U) \subseteq f(V)$ in Y (true of functions in general). F then satisfies the properties of the function in definition 3. It follows from bijectivity of f that for open $W \subseteq Y$, $F^{-1}(W) = f^{-1}(W)$, so F^{-1} has the same properties, as f^{-1} is a homeomorphism. X and Y are therefore categorically isomorphic. □

This result shows that categorical isomorphism of topological spaces is a strictly weaker notion of equality than homeomorphism.

4. Homeomorphism of Categorically Isomorphic T_1 Spaces

Example 1. Consider a finite topological space X. Let Y be the space such that for every point $x \in X$ there are two points x_1, x_2 in Y, and for each open $U \subseteq X$, $U' = \{x_1, x_2 | x \in U\}$ is open in Y (and these are all of the open sets in Y). It can be checked that the map sending $U \subseteq X$ to $U' \subseteq Y$ is bijective and preserves subset inclusion, so X and Y are categorically isomorphic, yet they are not homeomorphic, as Y has twice as many points as X so there is no bijection between them.

As is demonstrated by this example, categorical isomorphism of topological spaces is not equivalent to homeomorphism, as points can be added to a space without changing the subset structure of the open sets if they are indistinguishable (at least by inclusion in open sets) from some other point in the space. It is natural then to consider spaces where this cannot happen; spaces that require distinct points to be separated in some way. As is shown below, the T_1 condition is sufficient, based on the fact that a topological space is T_1 if and only if all singleton subsets are closed (5).

Proposition 2. *Categorically isomorphic T_1 topological spaces are homeomorphic.*

Proof Let X and Y be T_1 and categorically isomorphic. There is then a bijective function F from the open sets of X to those of Y such that F and F^{-1} preserve subset relationships.

$U \subseteq X$ for all open U in X, so $F(U) \subseteq F(X)$ for all U. F is surjective, therefore $V \subset F(X)$ for all open V in Y, so $Y \subseteq F(X)$, hence $F(X) = Y$.

As X is T_1, all singleton sets are closed, so for each $x \in X$, $C_x = X - \{x\}$ is open. As the complement of a single point, if $C_x \subseteq U$ then U must be either C_x or X. $F(C_x)$ is then an open set in Y. If $F(C_x)$ is not the complement of a single point, then (as it cannot be Y as $C_x \neq X$ and F is injective) the complement of $F(C_x)$ has multiple points. The complement of each of those points contains $F(C_x)$, and is open as Y is T_1. F^{-1} then sends said complement to an open set in X containing C_x (as F^{-1} preserves subset relationships) but not equal to X, a contradiction. Therefore, $F(C_x)$ is the complement of a point in Y.

Define $f : X \to Y : x \mapsto Y - F(C_x)$. f is well defined as $Y - F(C_x)$ is a single point as shown above. f is injective because F is injective, since the complement of each point in X is sent by F to the complement of a point in Y, but no two open sets in X can be sent to the same open set in Y, so as points in Y can be determined by their complements, no two points in X are sent by f to the same point. The same argument as above applied to F^{-1} shows that for any $y \in Y$, $D_y = Y - \{y\}$ is sent by F^{-1} to C_x for some $x \in X$. Therefore, $F(C_x) = D_y$, so $f(x) = y$, hence f is surjective.

Let U be open in X. U can be written as $U = \bigcap_{x \in X-U} C_x$. As f is injective, it distributes over intersections, so $f(U) = f(\bigcap_{x \in X-U} C_x) = \bigcap_{x \in X-U} f(C_x)$. As f is bijective, $f(C_x) = D_{f(x)}$, so $f(U) = \bigcap_{x \in X-U} D_{f(x)}$. $U \subseteq C_x$ for all $x \in X - U$, so $F(U) \subseteq F(C_x) = D_{f(x)}$ for all such x. Therefore, $F(U) \subseteq \bigcap_{x \in X-U} D_{f(x)} = f(U)$. Consider

$y \in f(U) - F(U)$. As $y \not\in F(U)$, $F(U) \subseteq D_y$, so $F^{-1}(F(U)) = U \subseteq F^{-1}(D_y) = C_{f^{-1}(y)}$. Therefore, $f^{-1}(y) \in X - U$, so $f(U) = \bigcap_{x \in X-U} D_{f(x)} \subseteq D_{f(f^{-1}(y))} = D_y$. However, $y \in f(U)$, so $y \in D_y$, a contradiction. Therefore $f(U) - F(U)$ is empty, so as $F(U) \subseteq f(U)$, $F(U) = f(U)$. By the same argument, for open $V \subseteq Y$, $F^{-1}(V) = f^{-1}(V)$.

For open $V \subseteq Y$, $f^{-1}(V) = F^{-1}(V)$, which is open in X, so f is continuous. Similarly, for open $U \subseteq X$, $f(U) = F(U)$, which is open in Y, so f^{-1} is continuous. f was shown above to be bijective, so f is a homeomorphism between X and Y. □

5. Conclusion

Fundamental to category theory is the idea that most areas of mathematics use the same basic concepts to study a variety of different topics, often focusing on some structure added to a set. However, topological structure can be far more complicated than many algebraic structures, as elements of both the set and its power set must be considered. For instance, the entire field of algebraic topology is devoted to using simpler algebraic objects to study topological spaces. While many algebraic structures can be related to topological spaces, a natural structure to consider would be one derived from the definition of a topology itself; some meaningful algebraic structure on the open sets of the topology. Ideally, with such a structure, topological spaces with equivalent such structures could be related in some more traditional way, and algebraic ideas pertaining to the structure (means of combination, substructures, structure preserving functions, etc.) would translate to topological ideas. As previously suggested, a category is not necessarily the most natural such structure, but it illustrates how meaningful information about a topological space can be extracted by focusing on the open sets themselves and not the points they contain.

6. Sources

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