Topological Modular Forms

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The theory of topological modular forms, constructed as a generalization of modular forms from number theory, illustrate a beautiful connection between algebraic geometry and homotopy theory. These notes provide what I see as a shortest possible path to understanding what TMF is and why it is interesting to homotopy theorists, for those more familiar with homotopy theory than algebraic geometry or number theory. This background was (and remains) my own as I prepared for my "A exam" on this topic, so these notes are meant to be the road map I would have found most helpful.

Roughly following the first reference, course notes by Charles Rezk, I build up to an operational definition of the spectrum tmf by analogy with the theory of Weierstrass curves, then outline some computations of its homotopy groups, weaving in throughout motivation and intuition from homotopy theory.

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1. Motivation

Chromatic homotopy theory studies the structure of a particular category of commutative ring spectra by breaking it up into more tractable pieces, which can tell us things about the sphere spectrum which is initial among all ring spectra (even if it is not among the spctra we consider). In particular, the spectra E we work with here are even periodic, so $E^m \cong E^{m+2}$ for all $m \in \mathbb{Z}$ and $E^{2k+1} \cong 0$. For E an even periodic ring spectrum, we write E_0 for the ring $E_0(pt) \cong E^0(pt) \cong \pi_0(E)$.

The chromatic decomposition of these spectra is based on their associated *formal group laws.* A computation using the Atiyah-Hirzebruch spectral sequence shows that for E even periodic, $E^0(\mathbb{C}P^\infty) \cong E_0[[t]]$ and $E^0(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E_0[[x, y]]$ as rings. These isomorphisms are not canonical, determined by the choice of "coordinates" t, x, y, but all of our constructions will be independent of coordinates.

For γ the universal bundle over $\mathbb{C}P^\infty$, the bundle $\gamma \otimes \gamma$ over $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ classifies a map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$. We then have a homomorphism $E_0[[t]] \to E_0[[x, y]]$, determined by the image of t, a formal power series $F_E(x, y)$ called a *formal group law* over E_0. F_E behaves like an abelian group operation (associative, commutative, has 0 as unit) by the same properties of $\otimes$. Different choices of coordinates on E define distinct formal group laws, but they are related by isomorphisms of formal group laws. Rather than describe what that means, we can define the coordinate-free notion of *formal group.*

If $B \to E_0$ is a homomorphism of E_0-algebras with the preimage of 0 in B a nilpotent ideal $nil(B)$, then F_E defines an abelian group structure on that ideal where the "sum" of two elements x and y is $F_E(x, y)$. Nilpotence ensures that each $F(x, y)$ converges in $nil(B)$, and B is called an *adic* E_0-algebra. A formal group is a functor from adic E_0-algebras to abelian groups whose underlying functor to sets is isomorphic to nil. A particular choice of that natural isomorphism is suggestively called a coordinate, and a formal group with a choice of coordinate can be shown to provide the same information as a formal group law.

Passing to formal group laws F lets us more easily define invariants of spectra. Let $[0]_F(x) = x$ and $[k]_F(x) = F([k-1]_F(x), x)$, the n-times repeated application of F. For p a prime and F a group law over a field of characteristic p, $[p]_F(x) = px + ... = v_n x^{p^n} + ...$ This n is called the *height* of F, set to be 0 for fields of characteristic 0, and it is an invariant of formal groups, meaning it doesn't depend on choice of coordinate. In particular, the height (for given p) is an invariant of even periodic ring spectra with fields as their coefficient rings.

Heights say a lot about formal group laws over fields, as if the field is separably closed then height completely determines the isomorphism classes of formal groups. The situation is more complicated for general rings, but we can think of the height of a formal group law for a given p as the lowest n such that $[p]_F(x) = v_n x^{p^n} + ... \pmod p$. In this setting it matters whether v_n is invertible or not, and how the height varies over different primes, but the amount of detail given here should be enough to motivate what follows.

One of the goals of chromatic homotopy theory is to detect information about the homotopy groups of spheres. One can search for p^n-periodic elements by applying a "multiply by p" map like in the definition of height, and keeping track of how many of these operations it takes to send particular group elements to zero. The theory then goes further by breaking up more general cohomology theories according to these heights, independently of the different primes.

One way to do this is by constructing a "universal theory for height n", and examples of this at low heights are well understood. Rational homotopy theory models height 0, and complex topological K-theory models height 1. The following sections cover how elliptic curves are used to build a model for height 2, what "universal" means in this setting, and some computational tools used to verify that universality.

2. Weierstrass Equations

Elliptic curves are a particularly nice class of abelian group schemes, with connections to number theory and cryptography. The Weierstrass equations discussed here are convenient representations of elliptic curves, and in fact every elliptic curve has a *Weierstrass parameterization*, discussed below. All of these definitions are in [1, Section 9], and for more on this see [2, Chapter III].

Definition 1. Let R be a ring. A Weierstrass equation in affine coordinates over R is an equation of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

where $a_1, a_2, a_3, a_4, a_6 \in R$. Call this $F_a(x, y)$, where $a = (a_1, a_2, a_3, a_4, a_6)$.

(It will soon make sense that we use a_6 instead of a_5.) We write C_a for the curve in P^2_R given by solutions to $Z^3F_a(X/Z, Y/Z)$, where C_a contains solutions to the affine equation F_a along with a single point e at infinity. A coordinate transformation is then an isomorphism (of schemes) $P_R^2 \to P_R^2$ sending $C_{a'}$ to C_a and preserving e. This turns out to always have the following form:

Definition 2. A coordinate change between Weierstrass equations given by parameters a' and a consists of constants $r, s, t, \lambda, \lambda^{-1} \in R$ such that setting

$$x = \lambda^{-2}x' + r$$

$$y = \lambda^{-3}y' + \lambda^{-2}sx' + t$$

yields $\lambda^6 F_a(x, y) = F_{a'}(x', y')$. Call this $\phi_{r,s,t,\lambda} : F_{a'} \to F_a$.

For fixed a_i, r, s, t, λ, a' is determined by

$a'_1 = \lambda(a_1 + 2s)$,

$a'_2 = \lambda^2(a_2 - a_1 s - s^2 + 3r)$,

$a'_3 = \lambda^3(a_3 + a_1 r + 2t)$,

$a'_4 = \lambda^4(a_4 - a_3 s + 2a_2 r - a_1 t - a_1 rs - 2st + 3r^2)$, and

$a'_6 = \lambda^6(a_6 + a_4 r - a_3 t + a_2 r^2 - a_1 rt + r^3 - t^2)$.

and a'_i will refer to these formulas throughout these notes. We want to treat these transformations as isomorphisms, or reparameterizations of the same curve, so they ought to be composable and invertible. Coordinate changes can be composed by $\phi_{r,s,t,\lambda} \circ \phi_{r',s',t',\lambda'} = \phi_{\nabla_{rr'},\nabla_{ss'},\nabla_{tt'},\nabla_{\lambda\lambda'}}$ where

$\nabla_{rr'} = r + \lambda^{-2}r'$,

$\nabla_{ss'} = s + \lambda^{-1}s'$,

$\nabla_{tt'} = t + \lambda^{-1}t' + \lambda^{-2}sr'$, and

$\nabla_{\lambda\lambda'} = \lambda\lambda'$.

Similarly we can compute formulas for the inverse coordinate transformation with parameters $\chi_r, \chi_s, \chi_t, \chi_\lambda$ that solve $\nabla_{x\chi_x} = 1$ for $x = r, s, t, \lambda$. Clearly $\chi_\lambda = \lambda^{-1}$, and we can solve for χ_r, χ_s, χ_t in the following matrix equation:

$$\begin{pmatrix} \lambda^{-2} & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ s & 0 & \lambda^{-3} \end{pmatrix} \begin{pmatrix} \chi_r \\ \chi_s \\ \chi_t \end{pmatrix} = \begin{pmatrix} 1 - r \\ 1 - s \\ 1 - t \end{pmatrix}$$

$$\begin{pmatrix} \chi_r \\ \chi_s \\ \chi_t \end{pmatrix} = \begin{pmatrix} \lambda^2(1-r) \\ \lambda(1-s) \\ \lambda^3(1-t-\lambda^2 s(1-r)) \end{pmatrix}$$

where the other inverse identity holds as well. With these compositions and inverses, along with identities where $(r, s, t, \lambda) = (0, 0, 0, 1)$, Weierstrass equations and coordinate transformations over R can be shown to form a groupoid.

3. Universality and Hopf Algebroids

One of the nice things about Weierstrass curves is that they can be uniquely characterized as pullbacks of a single "universal Weierstrass curve" over a ring A. Over a general ring R, a Weierstrass equation is determined by five elements $a_i \in R$, so a good choice of A would be the universal ring with five chosen elements, $A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$. In particular, we have $Hom(A, R) \cong R^5 \cong \{\text{Weierstrass equations over } R\}$. In terms of curves, any C_a is given by a pullback square

$$\begin{array}{ccc} C_a & \longrightarrow & \mathbf{C_a} \\ \downarrow & & \downarrow \\ Spec(R) & \longrightarrow & Spec(A) \end{array}$$

where $\mathbf{C_a}$ is the *universal Weierstrass curve* with $\mathbf{a_i} = a_i \in \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$. There is also a universal coordinate transformation, but it requires freely adding new parameters to the ring. $\Gamma = A[r, s, t, \lambda, \lambda^{-1}]$ has a Weierstrass curve with $a_i = a_i$ and a transformation into it with $r = r$ etc. from the curve with $a'_i = a'_i(a)$ according to the formulas for a'_i from the previous section. Any coordinate transformation of Weierstrass curves is determined by choices of a_i, r, s, t, λ, and a map from $\Gamma \to R$ picks out precisely that, so w have $Hom(\Gamma, R) \cong \{\text{coordinate transformations over } R\}$.

Both the object part and the morphism part of the functor $W : Ring \to Groupoid$ sending R to its groupoid of Weierstrass curves and coordinate transformations are representable, by A and Γ respectively. But more than that, the entire functor is represented by structure on A and Γ. We have the following diagram of rings:

$$A \;\begin{array}{c} \xrightarrow{\;t(a_i)=a_i\;} \\ \xleftarrow{\;\epsilon(r,s,t,\lambda)=(0,0,0,1)\;} \\ \xrightarrow[\;s(a_i)=a'_i\;]{} \end{array}\; \Gamma \xrightarrow{\;\nabla(r,s,t,\lambda)=(\nabla_{rr'},\nabla_{ss'},\nabla_{tt'},\nabla_{\lambda\lambda'})\;} \Gamma \otimes_A^{t,s} \Gamma$$

along with the similarly defined involution $\chi : \Gamma \to \Gamma$. Here r', r (etc.) are the variables in the left,right copies of Γ respectively in the tensor product. For each ring R, $Hom(-, R)$ sends colimits to limits, so $Hom(\Gamma \otimes_A^{t,s} \Gamma, R) \cong Hom(\Gamma, R) \times_{Hom(A,R)}^{s^*,t^*}$. The source, target, identity, composition, and inverse maps of the groupoid $W(R)$ are precomposition with, respectively, $s, t, \epsilon, \nabla, \chi$. In this sense the diagram above represents W.

Definition 3. A *Hopf algebroid* is a groupoid object in the category $Ring^{op}$, or equivalently a co-groupoid object in $Ring$.

The diagram above is a Hopf algebroid called (A, Γ), once it's been checked to satisfy the usual unit, associativity, and inverse equations (these aren't hard to do but would take up a lot of space), and any Hopf algebroid represents a functor $Ring \to Groupoid$ in the same way. Ignoring χ for the moment, an equivalent way of expressing the unit and associativity equations is by demanding that the diagram pictured above extends to a cosimplicial ring $\Delta \to Ring : n \mapsto \Gamma^{\otimes n}$ for $n > 0$:

$$A \;\begin{array}{c} \xrightarrow{\;t\;} \\ \xleftarrow{\;\epsilon\;} \\ \xrightarrow{\;s\;} \end{array}\; \Gamma \;\begin{array}{c} \longrightarrow \\ \longleftarrow \\ \xrightarrow{\;\nabla\;} \\ \longleftarrow \\ \longrightarrow \end{array}\; \Gamma \otimes_A^{t,s} \Gamma \;\begin{array}{c} \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{array}\; \Gamma^{\otimes 3} \qquad \cdots$$

The outermost face maps past s, t in lowest dimension are the canonical inclusions into (iterated) pushouts. The degeneracies and inner face maps are tensor products of respectively ϵ and ∇ with identities on all other copies of Γ. For instance, associativity means that $(\nabla \otimes id) \circ \nabla = (id \otimes \nabla) \circ \nabla$, which corresponds to the cosimplicial identity $d^1 \circ d^1 = d^2 \circ d^1$. So, very generally, a diagram like the above is a co-category object when it extends to a cosimplicial object of iterated pushouts. We can say something similar for co-groupoid objects by replacing Δ by its "groupoidal analogue" $\tilde{\Delta}$ which adds an involution to every object in Δ except 0, but that won't be needed that here.

This perspective is useful for algebraic topology, since by the Dold-Kan correspondence a Hopf algebroid viewed as a cosimplicial ring has an associated cohomologically graded complex of rings, which for the motivating example we write $C^\bullet(A, \Gamma)$ and call the *cobar complex.*

4. Elliptic Spectra

At this point, we have a rather precise sense of the structure of Weierstrass curves: over each ring they form a groupoid, and there is a universal curve over the object ring of the representing Hopf algebroid. For chromatic homotopy theory, the goal is to study even periodic ring spectra, and Weierstrass curves can be used to control the heights of these spectra.

Definition 4. An elliptic spectrum consists of an even periodic ring spectrum E, a pointed abelian group scheme C over E_0, and an isomorphism between the formal groups associated to E and C.

A pointed scheme over E_0 is a pair of maps of schemes $Spec(E_0) \xrightarrow{e} C \to Spec(E_0)$ that compose to the identity on $Spec(E_0)$. When C is affine it is *spec* of an adic E_0-algebra. C is a pointed abelian group scheme when it has a structure map $C \times_{Spec(E_0)} C \to C$ over $Spec(E_0)$ which, along with e, makes C an abelian group object in the category of pointed schemes over E_0, denoted $E_0/Scheme/E_0$.

For C as in the definition, there is a functor $\hat{C} : adic(E_0) \to Ab$ sending B to $Hom_{E_0/Scheme/E_0}(Spec(B), C)$. We want this to be a formal group to use in defining elliptic spectra, meaning its underlying functor to $\mathcal{S}et$ is isomorphic to nil sending B to the kernel of $B \to E_0$.

Proposition 5. *If* $e : Spec(E_0) \to C$ *factors as* $Spec(E_0) \xrightarrow{Spec(f)} Spec(R) \hookrightarrow C$*, then* R *is an* E_0*-algebra over* E_0 *and* $\hat{C}(B) \cong Hom_{alg(E_0)/E_0}(R, B)$*. Furthermore for* $I = ker(f)$*,* $Hom_{alg(E_0)/E_0}(R, B) \cong colim_n Hom_{adic(E_0)}(R/I^n, B)$*.*

For the latter statement, observe that any map in $Hom_{alg(E_0)/E_0}(R, B)$ sends I to the kernel of B, which is nilpotent, so the map factors through R/I^n for some n. This reduces the task of showing $\hat{C} \cong nil$ (as functors to $\mathcal{S}et$) to finding such an R where each map in $Hom_{alg(E_0)/E_0}(R, B)$ uniquely determines an element of $nil(B)$. Ideally R could be chosen isomorphic to $E_0[[t]]$, and if so then $\hat{C}$ is a formal group. However, instead of further investigating when this occurs, we will restate the definition of elliptic spectrum to assume this concern away:

Definition 6. An elliptic spectrum consists of an even periodic ring spectrum E, a pointed abelian group scheme C over E_0, and a natural isomorphism of functors $F_E \cong \hat{C} : adic(E_0) \to Ab$.

Example 7. Two familiar even periodic theories are ordinary periodic cohomology $HP^*(-, R) : X \mapsto H^*(X) \otimes_R R[u, u^{-1}]$ (u in degree 2) and complex K-theory, which have respectively the additive and multiplicative formal group laws $F_{HP}(x, y) = x+y$, $F_K(x, y) = xy+x+y$. These form elliptic spectra when paired with: the affine line A^1 which has associated to it the additive formal group, and the punctured line $A^1 - \{0\}$ with the multiplicative formal group.

The purpose of elliptic spectra is to restrict to spectra whose formal groups can be analyzed using algebraic geometry, where the work has often already been done. In particular we will restrict to spectra with formal groups arising from Weierstrass curves.

5. Weierstrass Parameterizations

Since the goal is to use Weierstrass curves to control the formal group laws of our spectra, we need to know how to detect Weierstrass curves among abelian group schemes. To avoid any coordinate dependence, we consider all possible presentations of a scheme as a Weierstrass curve as follows.

Definition 8. For C over $spec(R)$, a Weierstrass parameterization of C is an embedding $C \hookrightarrow P^2_R \times spec(R)$ over $spec(R)$ whose image in P^2 is a Weierstrass curve. Denote the set of such parameterizations $W(C/R)$.

Example 9. If C is given by a Weierstrass curve C_a over R, then $W(C/R) \cong \{(r, s, t, \lambda, \lambda^{-1})\}$ (values in R) is the set of coordinate transformations of C_a, but this isomorphism depends on the choice of parameters a, as the different parameterizations in $W(C/R)$ are given by applying each r, s, t, λ to a.

This suggests that when C has any Weierstrass parameterization then $W(C/R)$ is isomorphic to $Hom_{Ring}(\mathbb{Z}[r, s, t, \lambda^{\pm}], R) \cong Hom_{alg(R)}(R[r, s, t, \lambda^{\pm}], R)$. This turns out to be the case even as the level of elliptic spectra.

Proposition 10. *([1], 12.4)*

1. *There exists a ring spectrum Y such that for (E, C) an elliptic spectrum,*

$$hom_{alg(E_0)}(\pi_0(E \wedge Y), R) \xrightarrow{\cong} W(C \times_{spec(E_0)} spec(R)/R) =: W(C \otimes_{E_0} R)$$

2. *$(E \wedge Y, C \otimes_{E_0} E_0(Y) =: C')$ is an elliptic spectrum and any Weierstrass parameterization of C determines an isomorphism $\pi_0(E \wedge Y) \cong E_0[r, s, t, \lambda, \lambda^{-1}]$.*

3. *C' has a canonical Weierstrass parameterization represented by*

$$id \in hom_{alg(E_0)}(\pi_0(E \wedge Y), \pi_0(E \wedge Y))$$

Y can be constructed as the Thom spectrum of the map $\Omega U(4) \to \Omega U \cong \mathbb{Z} \times BU$, where the Thom spectrum of BU represents coordinates in the same way Y represents Weierstrass parameterizations, but we defer the proof to ([1], Sections 6 and 12).

In the groupoid of Weierstrass curves over E_0 there can be nontrivial automorphisms, i.e. instances where $a' = a$ but $(r, s, t, \lambda) \neq (0, 0, 0, 1)$, including in the connected component of that groupoid (if any) whose morphisms are described by $W(C/E_0)$. The group of automorphisms at any object determines a connected groupoid up to equivalence, so in a sense these symmetries describe all of the "Weierstrass structure" of C. Passing from (E, C) to $(E \wedge Y, C')$ adds a canonical Weierstrass parameterization, but ultimately destroys information, as the scheme C' equipped with a choice of parameterization no longer has nontrivial automorphisms, which would not be natural with respect to the canonical embedding.

Ultimately, our interest in describing C as a Weierstrass curve, and modifying (E, C) to make it one if necessary, comes back to heights of formal group laws. A Weierstrass curve C is always an abelian group scheme, so it makes sense to study its associated formal group $\hat{C}$.

Proposition 11. *([1],11.5 and [2],III.6.2) For C a Weierstrass curve over a field of positive characteristic, $\hat{C}$ has height 1, 2, or ∞.*

This is ultimately the motivation for using Weierstrass curves to describe the formal group laws of ring spectra. Their heights are constrained to 1, 2, or ∞ over fields, and while we won't go into more detail over general rings this suggests that these are good models generally for formal groups restricted to these heights. We can also specify that the curves with height ∞ are those isomorphic to that given by $a = (0, 0, 0, 0, 0)$ describing $(y^2 = x^3)$, and the curves with height 1 are isomorphic to $y^2 + xy = x^3$. All other curves are called *supersingular* and have height 2.

6. Higher Universality

In the same way that Weierstrass curves have a "universal model" in the curve over A with $a_i = a_i$, we would like to find a similar universal model of elliptic spectra. Ultimately this will be possible with certain restrictions, but first we begin at the level of Hopf algebroids:

Proposition 12. *([1], 14.2) For (E, C) an elliptic spectrum and Y as above, $(\pi_0(E \wedge Y), \pi_0(E \wedge Y^{\wedge 2}))$ forms a Hopf algebroid with a canonical map from (A, Γ).*

Proof. By the above proposition if we fix a Weierstrass parameterization of E landing in C_a, we have $\pi_0(E\wedge Y)\cong\pi_0(E)[r,s,t,\lambda^{\pm}]$ and $\pi_0(E\wedge Y^{\wedge 2})\cong\pi_0(E)[r,s,t,\lambda^{\pm},r',s',t',\lambda']$ and so on, with a copy of r,s,t,λ for each power of Y. The sets represented by these rings with respect to R are $W(C\otimes_{E_0}R)$ and $W(C\otimes_{E_0}R)^2$ respectively, with source and target maps defined by projection, identity by diagonal, and composition by ignoring the middle component of a triple of three choices of (r,s,t,λ) in R. In other words these rings represent the contractible groupoid on the object set $W(C\otimes_{E_0}R)$. These natural structure maps are all representable as ring homomorphisms, making up the Hopf algebroid structure.

Since $(E\wedge Y,C')$ has a canonical Weierstrass parameterization as C_a, there is a canonical choice of map $A\to\pi_0(E\wedge Y)$ sending the formal $a_i\in A$ to this a_i, and similarly a map from Γ picking out the formal transformation given by r',s',t',λ' in $\pi_0(E\wedge Y^{\wedge 2})$. Note that these maps commute with the map of Hopf algebroids induced by any map of elliptic spectra. □

This tells us that (A,Γ) is universal among not just rings with Weierstrass curves (and coordinate transformations), but also all Hopf algebroids of this sort describing canonically parameterized elliptic spectra in the sense of the previous section. This raises the question of whether this universality extends to elliptic spectra: is there some such "universal" parameterized elliptic spectrum with (A,Γ) as its corresponding Hopf algebra?

This is certainly not the case, as the groupoids represented by (A,Γ) are generally not contractible, in fact not even connected: A describes all Weierstrass equations on a ring, even non-isomorphic ones, not just how to get to others from a fixed choice of curve. So if we want to get anything close to this we must give up the expectation that this universal spectrum can itself be elliptic. And indeed, among ring spectra without assuming even periodicity, we can find something resembling a universal parameterized elliptic spectrum:

Theorem 13. *([1], 14.5) There exists a commutative ring spectrum tmf such that*

$$(\pi_0(tmf\wedge Y),\pi_0(tmf\wedge Y^{\wedge 2}))\cong(A,\Gamma)$$

Exciting as this may be, it would be *really* nice if the canonical maps $\pi_0(tmf\wedge Y)\to\pi_0(E\wedge Y)$ were induced by canonical maps $tmf\to E$, making tmf truly universal among elliptic spectra. This as it turns out is too much to ask, but the rest of these notes are devoted to showing that it almost works if we restrict to rings containing $\frac{1}{6}$. This suffices for many chromatic homotopy theory applications as a universal model for behavior of spectra at height 2, and the difficulties that arise when 3 or 2 is not invertible are interesting in their own right, but rather more complicated than the computation in the following sections.

7. Gradings and the Spectral Sequence

For the computations to come, we will need to move into the setting of bigraded rings. Recall there is a complex of rings $C^\bullet(A,\Gamma)$ associated to any Hopf algebra (A,Γ) by means of the corresponding cosimplicial ring, and without any extra work (the same arguments apply) we can restate the results of the previous section in terms of cobar complexes:

Proposition 14. *For (E,C) an elliptic spectrum, the cosimplicial ring $\pi_0(E \wedge Y^{\wedge(\bullet+1)})$ forms a Hopf algebroid with a canonical map of complexes from $C^\bullet(A,\Gamma)$.*

Theorem 15. *There exists a commutative ring spectrum tmf such that*

$$C^\bullet \pi_0(tmf \wedge Y^{\wedge(\bullet+1)}) \cong C^\bullet(A,\Gamma)$$

Here we haven't added anything new, but in both results we can make a stronger statement using π_* in place of π_0. First however, we must define a graded variant of (A,Γ).

Definition 16. $A_* = A[\eta,\eta^{-1}]$ and $\Gamma_* = \Gamma[\eta,\eta^{-1}]$ are graded rings with η in degree 2 and all of A and Γ in degree 0. To make (A_*,Γ_*) a graded Hopf algebroid (co-groupoid object in graded rings) we must also specify that $s(\eta) = \lambda^{-1}\eta$, $t(\eta) = \eta$, and all other structure maps preserve η.

η represents what is called the *invariant 1-form* of a curve, which we can loosely think of in analogy with differential geometry as fixing a metric on a curve, where λ tracks how a coordinate change resizes with respect to that metric.

Now both of the previous results can be stated in bigraded form replacing π_0 with π_* and (A,Γ) with (A_*,Γ_*):

Proposition 17. *For (E,C) an elliptic spectrum, the cosimplicial graded ring $\pi_*(E \wedge Y^{\wedge(\bullet+1)})$ forms a graded Hopf algebroid with a canonical map of bigraded complexes from $C^\bullet(A_*,\Gamma_*)$.*

Theorem 18. *There exists a commutative ring spectrum tmf such that*

$$C^\bullet \pi_*(tmf \wedge Y^{\wedge(\bullet+1)}) \cong C^\bullet(A_*,\Gamma_*)$$

In this form, the theorem tells us how to compute $\pi_*(tmf)$ using the Bousfield-Kan spectral sequence: for a cosimplicial spectrum of the form $E \wedge Y^{\wedge(\bullet+1)}$, this sequence of bigraded complexes has $E_1^{s,t} = \pi_t(E \wedge Y^{\wedge(s+1)})$ and converges to $\pi_{t-s}E$. So to compute $\pi_* tmf$, we begin by computing the page $E_2^{s,t} = H^{s,t}(C^\bullet(A_*,\Gamma_*))$ (which we abbreviate as $H^{s,t}(A_*,\Gamma_*)$).

8. Computing $\pi_* tmf[\frac{1}{6}]$

We do not compute all of $\pi_* tmf$ (see [1] chapters 15-19 for a more complete calculation) but instead focus on $\pi_* tmf[\frac{1}{6}]$ using the Bousfield-Kan spectral sequence, which begins with finding $E_2^{s,t} = H^{s,t}(A_*[\frac{1}{6}], \Gamma_*[\frac{1}{6}])$. The strategy will be to reduce $(A_*[\frac{1}{6}], \Gamma_*[\frac{1}{6}])$ to simpler algebroids without changing its cohomology. This is made possible by the following "change of rings" theorem:

Theorem 19. *([1],15.2) If $f : (A, \Gamma) \to (A', \Gamma')$ is a map of Hopf algebroids such that $\Gamma' \cong A' \otimes_A^{f,t} \Gamma \otimes_A^{s,f} A'$ and there is a map g such that $A \xrightarrow{s} \Gamma \to A' \otimes_A^{f,t} \Gamma \xrightarrow{g} R$ is faithfully flat, then f is an isomorphism on cohomology.*

Recall that a map $A \to R$ is faithfully flat if it makes its codomain into a flat module and the functor $(R \otimes_A -)$ reflects 0. This theorem gives us everything we need to compute $H^{**}(A_*[\frac{1}{6}], \Gamma_*[\frac{1}{6}])$ by showing that both maps in the following diagram are isomorphisms on cohomology:

$$(A_*[\tfrac{1}{6}], \Gamma_*[\tfrac{1}{6}]) \xleftarrow{f} (\bar{A}_*[\tfrac{1}{6}], \bar{\Gamma}_*[\tfrac{1}{6}]) \xrightarrow{\iota} (\bar{C}_*, \bar{C}_*)$$

Beware that this $\bar{C}_*$ is named to be analogous to $\bar{A}_*$, as we will see below when it's defined, not to be confused with the scheme C of an elliptic spectrum.

Definition 20. Let $\bar{A}_* = \mathbb{Z}[\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, \bar{a}_6]$ be a graded ring with $\bar{a}_i$ in degree $2i$, and $\bar{\Gamma}_* = \bar{A}_*[\bar{r}, \bar{s}, \bar{t}]$ with $\bar{r}, \bar{s}, \bar{t}$ in degrees 4, 2, and 6 respectively.

$(\bar{A}_*, \bar{\Gamma}_*)$ includes into (A_*, Γ_*) by $\bar{a}_i \mapsto a_i \eta^i$, $\bar{r} \mapsto r\eta^2$, $\bar{s} \mapsto s\eta$, $\bar{t} \mapsto t\eta^3$. Giving $(\bar{A}_*, \bar{\Gamma}_*)$ the Hopf algebroid structure with the same formulas as (A, Γ) just with $\lambda = 1$, it is easy to check that this inclusion is a map of Hopf algebroids. In terms of represented groupoids, this Hopf algebroid describes the groupoid of curves with transformations that have $\lambda = 1$, or equivalently ones that preserve the invariant 1-form η.

Lemma 21. *([1], 15.7) The change of rings theorem applies to this inclusion f.*

Proof. $A_* \cong \bar{A}_*[\eta^{\pm}]$, so if $A_* \otimes_{\bar{A}_*} M = 0$, then for all $x \in M$ $\eta \otimes x = 0$ implies $x = 0$ as there is no $\bar{A}_* \in \bar{A}_* - \{0\}$ with $\bar{A}_* \eta = 0$, so f is faithfully flat. f being faithfully flat is enough to satisfy the second condition as

$$\bar{A}_* \xrightarrow{s} \bar{\Gamma}_* \to A_* \otimes^{f,t}_{\bar{A}_*} \bar{\Gamma}_* \xrightarrow{id \otimes \epsilon} A_* \otimes_{\bar{A}_*} \bar{A}_* \cong A_* = f$$

Now for the first condition we observe that

$$A_* \otimes^{f,t}_{\bar{A}_*} \bar{\Gamma}_* \otimes^{s,f}_{\bar{A}_*} A_* \xrightarrow{t \otimes f \otimes s} \Gamma_*$$

is an isomorphism since it sends $\eta \otimes 1 \otimes 1$ to η and $1 \otimes 1 \otimes \eta$ to $\lambda^{-1}\eta$, so the image hits all of $a_i, r, s, t, \lambda, \eta$ and it is injective on the two differing copies of η (and clearly injective on everything else which is just $\bar{\Gamma}_*$). □

We now have by the change of rings theorem that $H^{**}(A_*, \Gamma_*) \cong H^{**}(\bar{A}_*, \bar{\Gamma}_*)$, a useful reduction even when 6 is not inverted. The next step is to show that the change of rings theorem applies to a map expressing the canonical form of Weierstrass equations in rings containing $\frac{1}{6}$.

So far we have used the most general form of Weierstrass equations, but in most rings the equations can be simplified. If $1/2 \in A$, we can change coordinates from any equation given by $(a_1, a_2, a_3, a_4, a_6)$ to the form $(0, \frac{1}{4}b_2, 0, \frac{1}{2}b_4 x, \frac{1}{4}b_6)$ where $b_2 = 4a_2 + a_1^2$, $b_4 = 2a_4 + a_1 a_3$, and $b_6 = 4a_6 + a_3^2$. Even better, if $1/6 \in A$ then there is a further coordinate change to an equation given by $(0, 0, 0, -c_4/48, -c_6/864)$ for $c_4 = b_2^2 - 24b_4$ and $c_6 = -b_2^3 + 36b_2b_4 - 216b_6$. One can check that this coordinate change is given (non-canonically) by $r = \frac{1}{3}a_2 + \frac{1}{12}a_1^2$ $s = \frac{1}{2}a_1$, $t = \frac{1}{2}a_3$, $\lambda = 1$.

This last parameterization is nearly unique. By inspecting the formulas for a'_i above we see that for a coordinate transformation between Weierstrass equations of this form, $a_1 = a'_1 = 0$ tells us that $s = 0$. $a_2 = a'_2 = 0$ further implies that $r = 0$, and $a_3 = a'_3 = 0$ mandates that t is also 0, so such transformations only depend on λ. We can then define a Hopf algebroid representing the groupoids of these equations and transformations with $\lambda = 1$, which are all discrete.

Definition 22. Let $\bar{C}_* = \mathbb{Z}[\frac{1}{6}, \bar{c_4}, \bar{c_6}]$ with $\bar{c_i}$ in degree $2i$, and note that $(\bar{C}_*, \bar{C}_*)$ with all structure maps the identity forms a (discrete) Hopf algebroid.

We now consider the map $f : (\bar{A}_*[\frac{1}{6}], \bar{\Gamma}_*[\frac{1}{6}]) \to (\bar{C}_*, \bar{C}_*)$ sending $\bar{a_1}, \bar{a_2}, \bar{a_3}, \bar{r}, \bar{s}, \bar{t}$ to 0, $\bar{a_4}$ to $-\bar{c_4}/48$, and $\bar{a_6}$ to $-\bar{c_6}/864$.

Lemma 23. *([1], 15.13) The change of rings theorem applies to f.*

Proof.

$$\bar{C}_* \otimes^{f,t}_{\bar{A}_*} \bar{\Gamma}_* \otimes^{s,f}_{\bar{A}_*} \bar{C}_* \cong \bar{C}_*$$

as $\bar{a_1}, \bar{a_2}, \bar{a_3}, \bar{r}, \bar{s}, \bar{t} \in \bar{\Gamma}_*$ all go to zero in the ring on the left (trivially for $\bar{a_i}$, $i = 1, 2, 3$, and for $\bar{s}$ we have $1 \otimes \bar{a_i} \otimes 1 = 1 \otimes \bar{a_i}' \otimes 1 = 0$ so $1 \otimes \bar{s} \otimes 1 = 1 \otimes \frac{1}{2}(a_1' - a_1) \otimes 1$, likewise for $\bar{r}, \bar{t}$). For the second condition, we want to define g such that the following composite is the identity (which is faithfully flat):

$$\bar{A}_* \xrightarrow{s} \bar{\Gamma}_* \to \bar{C}_* \otimes^{f,t}_{\bar{A}_*} \bar{\Gamma}_* \xrightarrow{g} \bar{A}_*$$

The ring $\bar{C}_* \otimes^{f,t}_{\bar{A}_*} \bar{\Gamma}_*$ represents coordinate transformations with codomain of the form represented by C. The left map from $\bar{A}_*$ represents the domain parameters of those coordinate transformations, so specifying g amounts to identifying naturally for each general Weierstrass equation a coordinate change (with $\lambda = 1$) into the form represented by C. Such a coordinate transformation is given above by $\bar{r} \mapsto \frac{1}{3}\bar{a_2} + \frac{1}{12}\bar{a_1}^2$ $\bar{s} \mapsto \frac{1}{2}\bar{a_1}$, $\bar{t} \mapsto \frac{1}{2}\bar{a_3}$. So the change of rings theorem applies. □

Having verified that both maps relating $(\bar{A}_*[\frac{1}{6}], \bar{\Gamma}_*[\frac{1}{6}])$ to $(A_*[\frac{1}{6}], \Gamma_*[\frac{1}{6}])$ and $(\bar{C}_*, \bar{C}_*)$ induce isomorphisms on cohomology, we can now prove the following:

Proposition 24. *$H^{s,*}(A_*[\frac{1}{6}], \Gamma_*[\frac{1}{6}]) \cong H^{s,*}(\bar{C}_*, \bar{C}_*) \cong \mathbb{Z}[\frac{1}{6}, \bar{c_4}, \bar{c_6}]$ for $s = 0$ ($|c_i| = 2i$) and is 0 for $s > 0$.*

Proof. The first isomorphism follows from the lemmas above and the change of rings theorem. For the second, observe that the cosimplicial graded ring of $(\bar{C}_*, \bar{C}_*)$ has the form

$$\bar{C}_* \rightrightarrows \bar{C}_* \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \bar{C}_* \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \cdots$$

with all maps the identity, so by Dold-Kan the associated complex has differentials alternating between 0 and the identity, starting with 0, and as such $H^{0,*}(A_*[\frac{1}{6}], \Gamma_*[\frac{1}{6}]) \cong \bar{C}_*$ and all other cohomology is 0. □

In conclusion, this bigraded complex has all differentials zero, so the spectral sequence collapses at the E_2 page, completing the proof of the following.

Theorem 25. *$\pi_* tmf[\frac{1}{6}] \cong \mathbb{Z}[\frac{1}{6}, \bar{c_4}, \bar{c_6}]$.*

$\mathbb{Z}[\frac{1}{6}, \bar{c_4}, \bar{c_6}]$ hosts the universal Weierstrass curve among those over rings containing $\frac{1}{6}$. This makes $tmf[\frac{1}{6}]$ come quite close to realizing the universal property we were hoping for: for any elliptic spectrum (E, C) over such a ring with C a Weierstrass curve, we have a map $\pi_* tmf \to \pi_* E$ canonical up to rescaling by λ.

9. Conclusion

Topological modular forms are a rich subject at the interface of algebraic geometry and homotopy theory, also with connections to number theory not covered here, which was for a long time the state of the art machinery for chromatic homotopy theory. There is so much beautiful mathematics that goes into defining tmf and placing it in a broader context that it would be impossible to fit into anything less than an entire book (which has been done), but I hope that in surveying the path taken in [1] with some additional narrative building I have provided an accessible and self contained introduction.

Here we didn't cover the construction of the spectrum tmf, other variants of tmf with slightly different universal properties, details of how cohomology theories are filtered by heights, other calculations one can do with tmf and what they mean, classical modular forms, and so much more that one can continue to learn about in the subject. Defining tmf by the kind of universality property we would like it to have, however, is enough to go even further in computations than we did here, and expresses an idea of the role the spectrum plays without the heavier algebraic geometry machinery that goes into its concrete definition. As someone primarily interested in the homotopy theoretic side of things, I found this approach very helpful.

10. References

1. Charles Rezk. Course Notes [Link].

2. Joseph Silverman. *The Arithmetic of Elliptic Curves.*

3. Aaron Mazel-Gee. *You Could've Invented tmf* [Slides].