Nerves of Higher Categories

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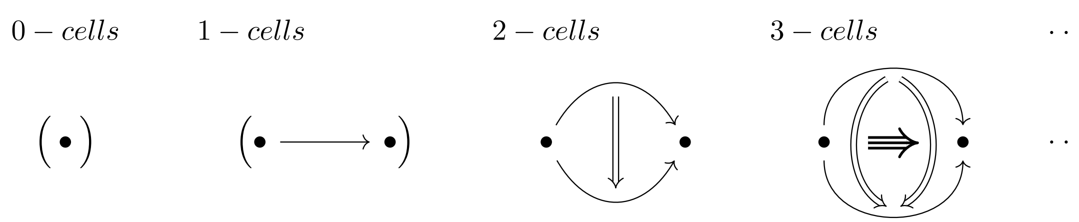
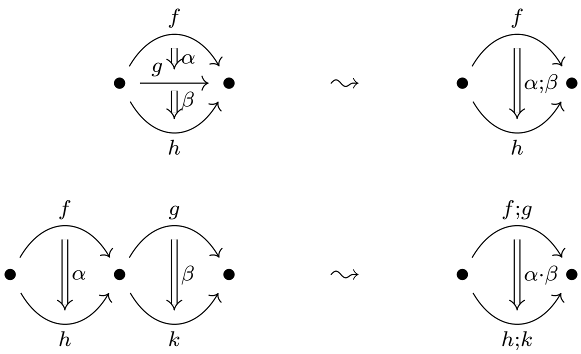
The nerve of a category is among the most fundamental constructions in category theory for homotopy theorists, to the point that it is popular to identify a category with its nerve, a simplicial set. This identification is the foundation for the geometric interpretation of $(\infty, 1)$-categories as quasicategories, simplicial sets that look like nerves of categories but where composites of arrows are no longer unique. Unlike more traditional algebraic notions of higher categories, quasicategories form the class of fibrant-cofibrant objects in a model category structure, and can be easily shown to have lots of convenient properties. While I prefer to think of the nerve not as an inclusion but as converting algebraic information to geometric, the geometric perspective has been incredibly fruitful in studying higher categories. In these notes I describe how analogous nerve operations can be constructed for n-categories and how those constructions extend to models for (∞, n)-categories.

1. Higher Categories

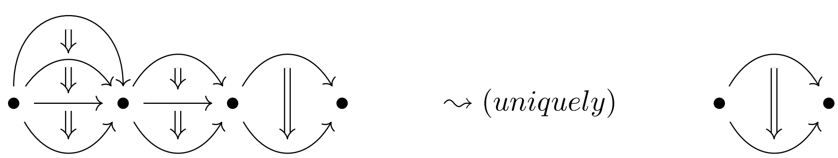
Often in mathematics, we deal with categories whose morphisms are related to eachother. Functions on ordered sets can be partially ordered, functors can be related by natural transformations, and continuous functions are related by homotopies. We want a categorical structure that can capture this kind of higher dimensional information. As such, higher categories allow us to describe settings where there are "higher dimensional" arrows between the morphisms, and arrows between those arrows, and so on. These higher dimensional arrows are called n-cells in dimension n, and look like:



Now we want n-categories to have identities and composition of n-cells, but they also need to play nicely with composition of lower dimensional cells. So for 2-cells we have two different kinds of composition:



The first, called vertical composition, is the natural way one would think of composing 2-cells. The second, called horizontal composition, can be thought of as composition of 1-cells acting on 2-cells. Both of these compositions have identities: for vertical composition, there is an identity 2-cell at each 1-cell, and for horizontal composition, the identity 2-cell at the identity 1-cell at each 0-cell is an identity. They are also both associative, but not just individually for compositions of 3 2-cells in either direction, but for any diagram that can be composed by a sequence of horizontal and/or vertical compositions: any order of such compositions will yield the same result. A category with 2-cells and compositions and identities satisfying these properties is called a 2-category.

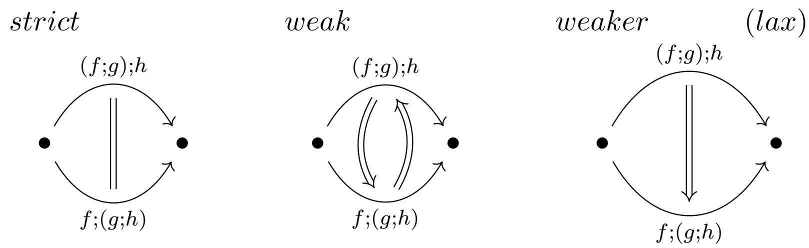


Example 1. The category of categories has 0-cells categories, 1-cells functors, and 2-cells natural transformations, with the usual horizontal and vertical compositions of natural transformations.

Similarly, an n-category has 1-cells up through n-cells (between parallel $(n-1)$-cells with shared boundaries), and the n-cells have n different compositions: one like vertical composition which composes a path of adjacent n-cells, then one induced by the composition of k-cells for $1 \leq k < n$. The composition operation for each k has identities given by the identity n-cell at each $(k-1)$-cell, and they are collectively associative in that any order of compositions of a diagram are equal. An ω-category has n-category structure for all n.

Example 2. The category of topological spaces has 1-cells continuous functions. It would be nice to be able to have 2-cells as homotopies, 3-cells homotopies between those homotopies, and so on. The problem is that vertical composition of homotopies is only associative up to higher homotopy, and this is true at all levels, so this is not a *strict* ω-category like those discussed above, but a *weak* ω-category, which is traditionally called an ∞-category.

This kind of weak higher categorical structure is very common, arguably more so than strict higher categories, and arises from the idea that when we have higher cells we can have properties that hold only "up to homotopy", where the higher cells play the role of homotopies. For 2-categories, properties like associativity come in many different strengths:



We will focus on the middle option, since it gives us properties up to higher cells with equalities weakened to equivalence, rather than simply arrows (which require careful choice of direction and is often weaker than necessary). A weak n-category is an n-category with all properties only required to hold weakly in this sense except for n-cells. A weak ω-category is called an ∞-category (this conflicts with modern usage somewhat as I describe below, but I think it is the most appropriate terminology).

Example 3. The category of topological spaces with continuous functions and (higher) homotopies is an ∞-category. However, we can observe that the n-cells for $n > 1$, all homotopies, are weakly invertible: the reverse of a homotopy composed with the original is *homotopic* to an identity. So here we have a weak inverse property in dimensions > 1. This is called an $(\infty, 1)$-category, which is unfortunately often all that people mean when they say ∞-category.

More generally, an (n, r)-category for $r \leq n$ is an n-category (let's assume everything from this point on is weak) with all k-morphisms for $k > r$ having weak inverses (for the vertical-like composition). An (n, n)-category is the same as an n-category, which includes (in my terminology) the case $n = \infty$. Note that this indexing is monotonic in the following sense: for $n < m$ any (n, r)-category can be seen as an (m, r)-category with only identity

k-morphisms for $n < k \leq m$, since these identities are certainly isomorphisms. Similarly, for $r < s$ an (n, r)-category can be seen as an (n, s)-category by forgetting the (weak) invertibility of the k-morphisms for $r < k \leq s$.

$$\begin{array}{ccc} (n,r)\text{-Categories} & \hookrightarrow & (n,s)\text{-Categories} \\ \downarrow & & \downarrow \\ (m,r)\text{-Categories} & \hookrightarrow & (m,s)\text{-Categories} \end{array}$$

One familiar with the modern usage of the term ∞-category will know that my notational departure from the new norm runs even deeper than I have admitted thus far. Rarely does a working mathematician talk about ∞-categories in terms of a sequence of composition operations satisfying weak associative and unital properties, even with the assumption that all morphisms above dimension 1 have weak inverses. They use the term to describe quasicategories (or occasionally other similar structures), which are just a kind of simplicial set that I describe below. This represents a fundamental distinction from how I (and perhaps others, I can only hope) think about categorical structures. My picture of categories is entirely algebraic, in the sense I have described, and is the definitional generalization of n-categories, which always refer to the algebraic gadgets I have defined. However, a homotopy theorist or algebraic geometer may find cumbersome all of these compositions and properties which carry their higher cell witnesses as extra data. They care about being able to write or cite proofs of properties of these things, and so far those proofs have been found for quasicategories, which can be seen as an 'equivalent' way of thinking about $(\infty, 1)$-categories. The meaning of this 'equivalence' is rather sketchy, but I will give one possible explanation for it below, and making it less sketchy is a research goal of mine.

2. The Nerve

Consider the functor $i : \Delta \to \mathcal{C}at : \bar{n} := \{0, ..., n\} \mapsto (0 \to 1 \to ... \to n)$. This functor is fully faithful, and we can consider it as an inclusion. Accordingly, I will write $\bar{n}$ to denote the corresponding category.

Definition 4. The nerve of a category $\mathcal{C}$ is the simplicial set $\mathcal{NC}$ with $\mathcal{NC}_n = Hom_{\mathcal{C}at}(\bar{n}, \mathcal{C})$, where for each map $f : \bar{n} \to \bar{m}$ in Δ, $\mathcal{NC}_f : Hom_{\mathcal{C}at}(\bar{m}, \mathcal{C}) \xrightarrow{f^*} Hom_{\mathcal{C}at}(\bar{n}, \mathcal{C})$.

Example 5. It quickly follows that $\mathcal{N}\bar{n} = \Delta^n$, the free n-simplex. The nerve of the category generated by $\cdot \leftarrow \cdot \rightarrow \cdot \rightarrow \cdot \leftarrow \cdot$ is a triangle with two edges attached to its first and last vertices.

These examples begin to illustrate how the nerve takes algebraic composition information and converts it to geometric information: composites look like triangles, and similarly longer composites look like higher simplices. While this construction gives a compelling visualization of the algebraic structure of a category, if we want to do this sort of thing for different algebraic structures we will need a way of analyzing what it means to represent algebraic structure geometrically. Fortunately, there is a definition of categories as algebras for a monad that provides both the algebraic and geometric interpretations canonically.

3. Nerves to Quasicategories

It is a nice exercise to prove that the nerve functor is fully faithful, meaning that the data of a functor between two categories is exactly the same as that of a simplicial map between their nerves. This means that one *can* think of categories as a special class of simplicial sets, specifically those sSets X with the property that

$$Hom(\Delta^n, X) \to Hom(\cdot \xrightarrow{1} \cdots \xrightarrow{n} \cdot, X)$$

is a bijection (so every n-simplex comes from a sequence of composable 1-simplices).

Definition 6. Quasicategories are simplicial sets X where the map $Hom(\Delta^n, X) \to Hom(\cdot \xrightarrow{1} \cdots \xrightarrow{n} \cdot, X)$ need only be a surjection, with some extra coherence conditions.

This means that in a quasicategory, every composable sequence of 1-simplices fits into an n-simplex whose other edges can be thought of as the various composites of those arrows including the total composition of the string in the $0n$ edge. But this n-simplex is not unique, so we can think of a quasicategory as a category where strings of arrows can have more than one composite. However, one can prove that any two such composites are faces of some $(n+1)$-simplex, and any two such $(n+1)$-simplices are faces of some $(n+2)$-simplex and so on, so the composites form a contractible space in some sense. Quasicategories are called ∞-categories by many people (not me), because in many important ways they resemble the $(\infty, 1)$-categories I defined above.

The intuition for this is that unlike nerves of categories, the simplices in dimensions > 1 are not uniquely determined by their *spine* of composable edges, so they have their own meaning and can be thought of as n-cells (without a clear source and target, but the best guess would be between the formal composite of the arrows in the spine and the 0n arrow of the simplex). Associativity here seems quite strict, as in an n-simplex any order of compositions of the edges in the spine will ultimately reach the 0n edge, but the composition itself has become weak, where n 1-cells no longer have a unique composite but rather a guarantee of at least one n-cell describing a choice of consistent compositions.

It should not be obvious why this is equivalent to an algebraic weak $(\infty, 1)$-category, and an answer to this is a research interest of mine, but an informal justification can be given as follows. An algebraic $(\infty, 1)$-category can be seen as a category (weakly in some sense) enriched in $(\infty, 0)$-categories, since between any two objects there is an ∞-category with morphisms as objects where all cells are invertible since they come from cells above dimension 1 in an $(\infty, 1)$-category. These are called ∞-*groupoids*, and Grothendieck's *Homotopy Hypothesis* is the claim that these describe all the homotopy information of topological spaces. So from the perspective of homotopy theory, an $(\infty, 1)$-category is equivalent to a category enriched in topological spaces or simplicial sets. Simplicially enriched categories are known to be equivalent to quasicategories via the "Homotopy Coherent Nerve", which I will outsource to a reference below.

The punchline here is that simplicial sets can completely describe categories via the nerve, and relaxing the conditions for a simplicial set to be a category gives something widely considered to be equivalent to $(\infty, 1)$-categories. We will see that a similar construction works for $n > 1$, where we construct analogous nerves of strict n-categories, weaken the properties that define them, and get a way of modeling (∞, n)-categories.

4. A Monad Construction

Definition 7. Recall that a monad is a functor $M : \mathcal{C} \to \mathcal{C}$ along with natural transformations $\eta : Id_{\mathcal{C}} \to M$, $\mu : MM \to M$ such that the following diagrams commute:

$$\begin{array}{ccc} MMM & \xrightarrow{M\mu} & MM \\ {\scriptstyle \mu_M}\downarrow & & \downarrow{\scriptstyle \mu} \\ MM & \xrightarrow[\mu]{} & M \end{array} \qquad \begin{array}{ccccc} M & \xrightarrow{M\eta} & MM & \xleftarrow{\eta_M} & M \\ & \searrow\!\!= & \downarrow{\scriptstyle \mu} & =\!\!\swarrow & \\ & & M & & \end{array}$$

Definition 8. Further recall that an algebra A for M is a map $MA \xrightarrow{a} A$ such that the following diagrams commute:

$$\begin{array}{ccc} MMX & \xrightarrow{\mu_X} & MX \\ {\scriptstyle Ma}\downarrow & & \downarrow{\scriptstyle a} \\ MX & \xrightarrow[a]{} & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\eta_X} & MX \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle a} \\ & & X \end{array}$$

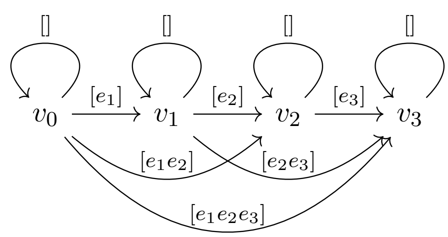
We can define categories as algebras for a monad on the category of (directed) graphs, where I take graphs to mean diagrams $E \rightrightarrows V$ in $\mathcal{S}et$. More precisely, note that the category of graphs is a contravariant functor (presheaf) category $\mathcal{S}et^{G^{op}} =: \hat{G}$ where G is the category $v \rightrightarrows^s_t e$. For a graph $X : G^{op} \to \mathcal{S}et$, we write X_v and X_e for the vertex and edge sets $X(v)$ and $X(e)$.

The monad we want to consider takes a graph X to the graph underlying the free category on X. The free category on a graph X has the vertices of X as objects, and has as morphisms paths in X of finite length (not necessarily nonzero). Composition is by path concatenation, and identities are the empty paths at each vertex.

Definition 9. Our monad, which I will call M, then sends the graph X to the graph whose vertices are the same and whose edges are finite length paths in X. The unit map $\eta_X : X \to MX$ is the identity on vertices and sends an edge in X to the length 1 path containing just that edge. The multiplication map $\mu_X : MMX \to MX$ takes a path in MX, which is a string of composable paths in X, and concatenates them into a single path. The monad laws follow from basic properties of path concatenation.

Example 10. Consider the graph $\tilde{3} = v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} v_3$.

$M\tilde{3} =$



Now we want to see categories as algebras for this monad. For a graph A with an algebra structure map $MA \xrightarrow{a} A$, a sends paths in A to edges in A, which we can think of as composition. The algebra law for the unit makes sure that a is the identity on vertices, so composition preserves source and target, and that a sends a length 1 path to the edge it contains. The law for multiplication makes sure that the composition of a concatenation of paths is the same as the composite of the compositions of each of those paths. This property implies that our composition is both associative and unital: if $[e_1 e_2 e_3]$ is a path of compatible edges in A then we have $a[a[e_1]a[e_2e_3]] = a[e_1e_2e_3] = a[a[e_1e_2]a[e_3]]$, and for e an edge in A with $[]$ the empty path at its source we also have $a[a[]a[e]] = a([]+[e]) = a[e] = e$ (and same for the empty path at the target of e). Thus an algebra for this monad is precisely a category.

Example 11. A nice simple example of an algebra for this monad is the map $MM\tilde{3} \xrightarrow{\mu_{\tilde{3}}} M\tilde{3}$, sending any path in $M\tilde{3}$, which is a path *of paths* in $\tilde{3}$, to its concatenated path. $MM\tilde{3}$ is infinite (as a path of paths can contain infinitely many empty paths), but this algebra map would send for instance both $[[e_1e_2]]$ and $[[e_1][e_2]]$ to $[e_1e_2]$, and send $[[e_1e_2][][e_3][]]$ to $[e_1e_2e_3]$.

Note that we can do this for $M\tilde{3}$ but not $\tilde{3}$, since there is no edge in $\tilde{3}$ for the path $[e_1e_2]$ (or any path of length $\neq 1$) to be sent to by an algebra map, since there is no edge from v_1 to v_3. In $M\tilde{3}$, any path of its edges has exactly one edge with the appropriate source and target, so the structure map $MM\tilde{3} \to M\tilde{3}$ is unique. In general however, there may be many possible M-algebra structure maps making a graph into a category, just as there can be exactly one or none at all.

Remark 12. This may seem rather meta: we are giving something like a definition of a category using such tools as a monad and the category of graphs, which already use the notion of categories in their definitions. This is one of the most beautiful aspects of category theory: that it can use its own techniques to describe itself. Like the analogous situation in set theory, the reason this is an acceptable thing to do is that we can take whatever definition of large categories you prefer as our meta-theory, and use that language to give this particular description of the theory of small categories. What we mean by 'large' and 'small' here is, again like set theory, rather flexible, but it suffices to accept that one can use the language of categories to define (or give an alternative characterization of) objects that behave exactly like categories themselves.

This is a nice way to think about categories because it clearly separates the different components of what makes up a category. The category of graphs describes the underlying *data* of a category, namely objects and morphisms. The functor M (via the algebra map a) describes the *structure* put on that data, namely the specification of composites and identities. The multiplication (via the multiplication property of a) describes the *properties* of the structure. The unit map η (via the unit property of a) describes how the structure acts on the underlying data. The combination of data, structure, and properties is a powerful perspective on how we define all mathematical things, and monads provide a way of expressing those things formally.

But how is any of this related to nerves? It turns out this monad can be redefined in a way that provides more insight:

Definition 13. I call a functor of the form $F : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$ *familially representable* if it sends X to $FX : c \mapsto \sqcup_{i\in I_c} Hom(Y_{c,i}, X)$ for some fixed $\{Y_{c,i}\}$ with structure maps given by precomposing with maps between the $\{Y_{c,i}\}$ in $\hat{\mathcal{C}}$ componentwise.

I claim that our monad M is familially representable. Let $\tilde{n}$ denote the graph with just n composable edges, $\tilde{0}$ being the trivial graph with one vertex and no edges. MX has the same vertices as X, which are given by $Hom(\tilde{0}, X)$. The edges of MX are finite paths in X, which can be partitioned according to their length, where a path of length n is the image of a map from $\tilde{n}$ to X. Therefore the edges of MX are given by $\sqcup_{n\in\mathbb{N}} Hom(\tilde{n}, X)$, and the sources and targets come from precomposing with the source/target maps $\tilde{0} \rightrightarrows \tilde{n}$.

$$\tilde{0} = \qquad v_0$$

$$\tilde{1} = \qquad v_0 \xrightarrow{e_1} v_1$$

$$\tilde{2} = \qquad v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2$$

$$\tilde{3} = \qquad v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} v_3$$

$$\tilde{4} = \qquad v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} v_3 \xrightarrow{e_4} v_4$$

We can now see how the algebraic structure of categories comes from the subcategory of graphs generated by $\{\tilde{n}\}$. These graphs contain the representable vertex $\tilde{0}$ and the representable edge $\tilde{1}$. Composition sends a copy of any of these graphs in an algebra A to an edge in A. And for associativity, we can see that these graphs are closed under what I call *shape composition*, which is taking one of these graphs $\tilde{n}$ and "plugging in" any n of these graphs to each of the n edges: the graph with n composable edges replaced with, respectively, the graphs of $m_1, .., m_n$ composable edges gives the graph of $\sum_i m_i$ edges. This corresponds to what we seen in the definition of μ.

Example 14. The graph $\tilde{4}$ admits a shape composition with, respectively, $\tilde{2}$, $\tilde{0}$, $\tilde{1}$, $\tilde{2}$ plugged in for its edges, and this composition yields the graph $\tilde{5}$.

$$\cdot \to \cdot \to \cdot \to \cdot \to \cdot$$

$$\cdot\Big(\to\cdot\to\Big)\cdot\Big(\Big)\cdot\Big(\to\Big)\cdot\Big(\to\cdot\to\Big)\cdot$$
$$\cdot\to\cdot\to\cdot\to\cdot\to\cdot\to\cdot$$

Remark 15. We can even further see that this subcategory is in fact generated by $\tilde{0}, \tilde{1}, \tilde{2}$ under shape composition, so it is the smallest such subcategory that can encode composition of two arrows.

Now that we've reduced this monad to looking at the subcategory spanned by the graphs $\tilde{n}$, we can show how this relates to nerves. Take the subcategory of M-algebras spanned by the algebras $MM\tilde{n} \xrightarrow{\mu} M\tilde{n}$. For a given n, this algebra is the category $\bar{n}$, so this subcategory is precisely Δ! The nerve then takes a category $\mathcal{C} = MA \xrightarrow{a} A$ to the simplicial set

$$\bar{n} \mapsto Hom_{\mathcal{C}at}(\bar{n}, \mathcal{C}) = Hom_{\mathcal{C}at}(M\tilde{n}, \mathcal{C}) \cong Hom_{\hat{G}}(\tilde{n}, A)$$

where the isomorphism on the right is that of the classical adjunction from a category with a monad to the category of algebras for the monad. So what the nerve does is separate out the different components of the piecewise structure of M, where we know that these form a simplicial set because the morphisms of the simplex category correspond to the maps on each component given by the algebra structure map a. So we can either describe this structure algebraically by an algebra map $MA \to A$, or geometrically by a simplicial structure extending A to the sets $Hom_{\hat{G}}(\tilde{n}, A)$.

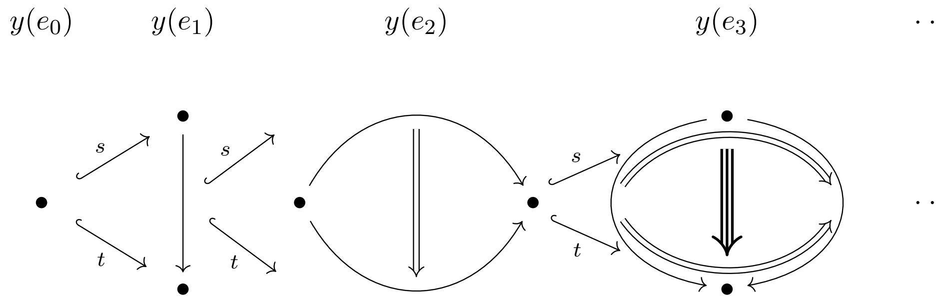
The intuition here is that an algebra map $MA \xrightarrow{a} A$ tells us how to extend any map $\tilde{n} \to A$ to a map $M\tilde{n} \to A$ by specifying where the composite paths are sent, so we can add extra structure maps between those sets corresponding to those composites, and also maps corresponding to composites within a composite of shapes. The category of $\tilde{n}$ graphs with these extra maps added is precisely Δ.

5. Generators for n-categories

We now want to imitate this construction to give us nerves of n-categories. First, we have to define what is the "data" of an n-category:

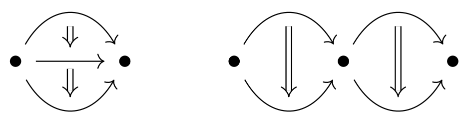
Definition 16. An n-graph (or n-globular set) is a functor $G_n^{op} \to \mathcal{S}et$, where G_n is the category generated by $e_0 \rightrightarrows_t^s e_1 \rightrightarrows_t^s e_2 \rightrightarrows \cdots \rightrightarrows_t^s e_n$ with $Hom(e_i, e_j) = \{s, t\}$ for $i < j$ (empty but for identities otherwise) and $s; f_1; ...; f_k = s$, $t; f_1; ...; f_k = $ t.

The representable functors $y(e_n)$ in $\hat{G}_n$ can be thought of as an n-cell, with the arrows pointing from s to t:



One can check that this category precisely describes the n-cells and their inclusions into one another. So an n-graph has vertices (0-cells), edges (1-cells), edges between edges (2-cells), and so on up to dimension n edges whose source and target must have the same source and target in each dimension. A monad construction like the one above then describes how to put an algebraic structure on an n-graph to make it an n-category.

The above description of the free category monad in terms of the subcategory of $\tilde{n}$ graphs tells us exactly how to proceed. We want our algebra maps to send each of the n "composition shapes" (as in ways of sticking composable n-cells together) like

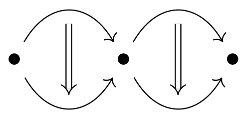


in some n-graph X to an n-cell in X in a way that satisfies similar unit and associativity rules. We will think of these, along with the representables, as generators for the representing n-graphs of our new monad.

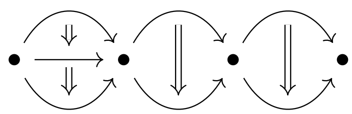
Definition 17. Let Z_n be the full subcategory of n-graphs consisting of those n-graphs generated by the representables and all composition shapes under shape composition.

Here one must pay careful attention to what shape composition means in this more general setting. While a precise definition can be formulated in terms of certain colimits indexed by categories of elements, it suffices for our purposes to say the following. Let $Z_{n,k}$ consist of those n-graphs in Z_n with cells only up to dimension k. Closure under shape composition means that any consistent way of plugging n-graphs in $Z_{n,k}$ into the k-cells of another n-graph in Z_n should give another n-graph in Z_n.

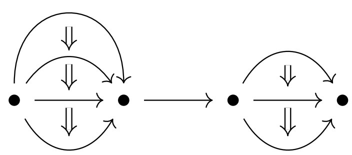
Example 18. Here are a few iterations of shape composition in Z_2:



Starting with a horizontal composite of two 2-cells, we can plug in a vertical composite into the first and a horizontal composite into the second to get another 2-graph in Z_2:



We can then plug another vertical composite into the upper leftmost 2-cell, a single 2-cell into the lower leftmost 2-cell, a 1-cell (thought of as a degenerate 2-cell) into the center 2-cell, and a vertical composite into the rightmost 2-cell:



This process can be repeated in any number of ways to give other 2-graphs in Z_2, but ultimately any such 2-graph would be a horizontal composite of j vertical composites of, respectively, $i_1, ..., i_j$ 2-cells where $i_k \in \mathbb{N}$. In the above example, for instance, $j = 3$, $i_1 = 3$, $i_2 = 0$, and $i_3 = 2$. It isn't immediately obvious why all shape compositions of these 2-graphs would be of this form, but one of the key reasons is that shape composition only allows for *consistent* choices of 2-graphs in Z_2 plugged into the 2-cells of a diagram. So starting with a vertical composite and trying to plug in a horizontal composite to the top 2-cell but a single 2-cell into the bottom one wouldn't work, since those two diagrams don't agree on the middle 1-cell of the vertical composite (the horizontal composite on top wants that 1-cell replaced with a composite and the single 2-cell on bottom wants it to be a single 1-cell). One could, however, plug horizontal composites into both the top and bottom 2-cells, since then the middle 1-cell could consistently be replaced by a composite. There is a similar characterization of the n-graphs in Z_n.

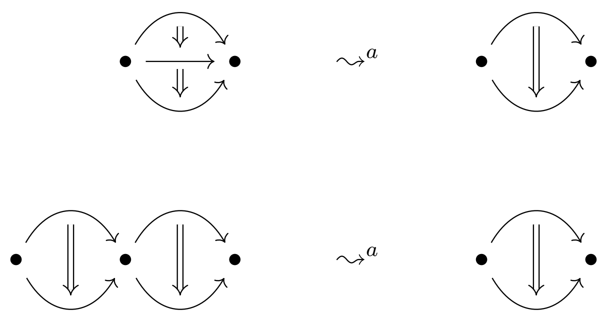
We can now define our monad on n-graphs using $\{\tilde{0}\} = Z_{n,0} \subset Z_{n,1} \subset ... \subset Z_{n,n} = Z_n$:

Definition 19. The free n-category monad $M_n : \hat{G}_n \to \hat{G}_n$ is given on an n-graph X by

$$M_n X_{e_k} = \sqcup_{z \in Z_{n,k}} Hom(z, X)$$

with unit defined by the inclusion of the representables in Z_n and multiplication defined using shape composition just like in the definition of M for $n = 1$ above.

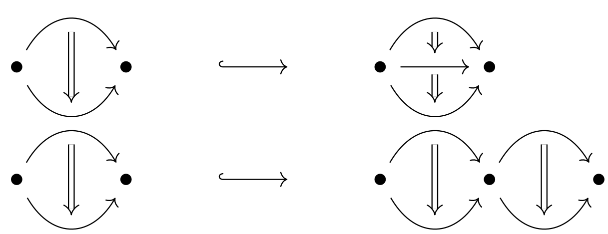
Algebras $M_nA \xrightarrow{a} A$ for this monad, like in the case of 1-categories, send each of these diagrams of (possibly identity) k-cells to a single k-cell, respecting the data and obeying a version of unit/associativity according to the structure of Z_n. These algebras are precisely the n-categories!



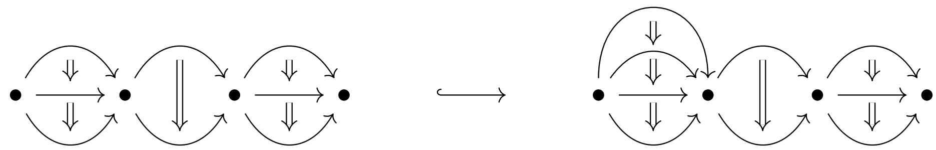
Then, just as we saw that the full subcategory consisting of M_1Z_1 is isomorphic to Δ, we will see that a natural nerve of n-categories takes values in the category M_nZ_n.

Definition 20. The category Θ_n is the full subcategory of n-categories consisting of those algebras of the form M_nz for some $z \in Z_n$.

As we discussed in the previous section for the case of $n = 1$, the subcategory Z_n of n-graphs only has simple inclusion maps between the n-graphs. Extending to Θ_n adds maps corresponding to composition: just as Δ has a map $\tilde{1} = M\tilde{1} = \bar{1} \to \bar{n} = M\tilde{n}$ identifying the composite of the n edges in $\tilde{n}$, Θ_n has a similar map from the n-cell to each composite of n-cells:



Here these inclusions should both be thought of as analogous to the map $\bar{1} \to \bar{2}$ in Δ sending the 1-simplex to the edge between the vertices labeled 0 and 2 in the 2-simplex. Again in analogy with Δ, there are more maps added to Z_n that look like the above maps applied inside a shape composition, corresponding to composing a single pair of n-cells within a larger diagram like below:



Here the analogy is with the maps $\bar{n} \to \overline{n+1}$ sending the $i, i+1$ edge in the n-simplex to the $i, i+2$ edge in the $n+1$ simplex. These added maps generally correspond to the inner face maps in Δ which for nerves of categories witness inner compositions within a string of morphisms, just as these maps witness inner composition in a diagram of n-cells. The degeneracy maps have analogues as well, with for each $k < m \leq n$ a map in Θ_n from the k-cell to the m-cell, and similarly to faces all instances of these maps within a larger shape composite. So Θ_n is in a sense (which can be made precise in various ways) a generalization of the simplex category to higher dimensional diagrams. This category is then the natural index for the functor category where a nerve of n-categories will land in.

Definition 21. For an n-category A, its n-nerve is the Θ_n-set $\mathcal{N}_n A$ given by $\mathcal{N}_n A = Hom_{\hat{G}_n}(z, A) \cong Hom_{n\mathcal{C}at}(M_n z, A)$.

Like the case of 1-categories, we have that for A an n-category, any diagram in the underlying graph of a shape z in Z_n extends by the M_n-algebra map to a diagram of shape $M_n z$, so we can define all of the desired structure maps on the sets $Hom_{\hat{G}_n}(z, A)$. So the nerve $\mathcal{N}_n$ of A consists of the set of diagrams of shape z in A, for all z in Z_n, and these sets of diagrams are related by all the maps in Θ_n. These maps correspond to inclusions of diagrams, compositions within those diagrams, and degeneracies of those diagrams, just like the maps in Δ describe 1-dimensional composable diagrams in this way. One can also show that this nerve functor $\mathcal{N}_n$ is fully faithful, by the same argument as in the case $n = 1$.

This all works for $n = \infty$ as well in exactly the same way just without the upper bound on dimension. Θ_∞ is simply called Θ, and each Θ_n is often identified with its inclusion into Θ.

6. Geometric (∞, n)-Categories from Nerves

We can now characterize nerves of n-categories just as we did for 1-categories:

Proposition 22. *A Θ_n-set X is the nerve of an n-category if and only if the map $Hom(M_n Y, X) \to Hom(Y, X)$ induced by the unit map on Y is a bijection for all Y in Z_n.*

This suggests that, in analogy with the definition of quasicategories, weakening this definition to only require those maps to be surjections (with extra conditions making them still equivalences) would give a reasonable geometric definition of (∞, n)-categories, called n-quasicategories, if one is sufficiently content that quasicategories model $(\infty, 1)$-categories. However, a result of Ara shows that this does not work perfectly when trying to construct a model category structure on Θ_n-sets for modeling (∞, n)-categories. A model structure can be defined just like the one for quasicategories, with these n-quasicategories as the fibrant cofibrant objects, but not all traditional equivalences of n-categories lift by the nerve to weak equivalences. These maps can be added to the class of weak equivalences to get a model structure that matches other models of (∞, n)-categories, but now not all nerves of n-categories are fibrant. This suggests that the relationship is more complicated than it may seem!

7. References

These notes are the product of my synthesis of a large number of sources over quite a bit of time, along with perspectives I've developed in the course of an ongoing research project. That said, here are some helpful sources on these and related topics:

- Tom Leinster on Higher Categories (Appendix C is on familial representability): Link
- Eugenia Cheng and Aaron Lauda on Weak Higher Categories (including Θ): Link
- Emily Riehl on Simplicial Sets: Link
- Clark Barwick and Chris Schommer-Pries on Strict n-Categories and Homotopy Theories of (∞, n)-Categories: Link

Additionally, the following are the promised references for topics mentioned in these notes without any real explanation:

- Emily Riehl on Quasicategories and the Homotopy Coherent Nerve: Link
- Dimitri Ara on Higher Quasicategories: Link

Lastly, here is a far more advanced paper that develops these ideas very deeply for Θ and a class of definitions of (algebraic) ∞-categories:

- Clemens Berger on Θ-sets, Nerves, Globular Operads, and Model Structures: Link