# 1 Topology, Topological Spaces, Bases

**Definition 1.** A *topology* on a set $X$ is a collection $\mathcal{T}$ of subsets of $X$ having the properties

- $\emptyset$ and $X$ are in $\mathcal{T}$.
- Arbitrary unions of elements of $\mathcal{T}$ are in $\mathcal{T}$.
- Finite intersections of elements of $\mathcal{T}$ are in $\mathcal{T}$.

$X$ is called a topological space. A subset $U$ of $X$ is called *open* if $U$ is contained in $\mathcal{T}$.

**Definition 2.** Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on $X$. If $\mathcal{T}' \supset \mathcal{T}$, then $\mathcal{T}'$ is said to be *finer* than $\mathcal{T}$. If $\mathcal{T}' \subset \mathcal{T}$, then $\mathcal{T}'$ is said to be *coarser* than $\mathcal{T}$. If either are true, $\mathcal{T}$ and $\mathcal{T}'$ are said to be *comparable.*

**Definition 3.** If $X$ is a set, a *basis* for a topology $\mathcal{T}$ on $X$ is a collection $\mathcal{B}$ of subsets of $X$ such that

- For each $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
- If $x \in B_1 \cap B_2$, where $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$

If $\mathcal{B}$ satisfies these, the *topology $\mathcal{T}$ generated by $\mathcal{B}$* is defined as follows: A subset $U$ of $X$ is said to be open in $X$ if for each $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$.

**Proposition 1.** *Let $X$ be a set; let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on $X$. Then $\mathcal{T}$ equals the collection of all unions of $\mathcal{B}$.*

**Definition 4.** The *standard topology on* $\mathbb{R}$ is the topology generated by a basis consisting of the collection of all open intervals of $\mathbb{R}$.

**Proposition 2.** *Let $X$ be a topological space. Suppose $\mathcal{C}$ is a collection of open sets of $X$ such that for each open set $U$ of $X$ and each $x \in U$, there is an element $C \in \mathcal{C}$ with $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of $X$.*

**Proposition 3.** *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on $X$. Then the following are equivalent:*

- *$\mathcal{T}'$ is finer than $\mathcal{T}$.*
- *For each $x \in X$ and each $B \in \mathcal{B}$ containing $x$, there is a basis element $B'$ such that $x \in B' \subset B$.*

**Definition 5.** A *subbasis* $\mathcal{S}$ for a topology on $X$ is a collection of subsets of $X$ whose union equals $X$. The *topology generated by the subbasis* $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

## 2 Product, Subspace, and Quotient Topologies

**Definition 6.** Let $X$ and $Y$ be topological spaces. The *product topology* on $X \times Y$ is the topology having a basis $\mathcal{B}$ that is the collection of all sets of the form $U \times V$, where $U$ is open in $X$ and $V$ is open in $Y$.

**Theorem 4.** *If $\mathcal{B}$ is a basis for the topology of $X$ and $\mathcal{C}$ is a basis for the topology of $Y$, then the collection $\mathcal{D} = \{B \times C | B \in \mathcal{B}$ and $C \in \mathcal{C}\}$ is a basis for the topology on $X \times Y$.*

**Definition 7.** Let $X$ be a topological space with topology $\mathcal{T}$. If $Y$ is a subset of $X$, the collection $\mathcal{T}_Y = \{Y \cap U | U \in \mathcal{T}\}$ is a topology on $Y$, called the *subspace topology*. With this topology, $Y$ is called a *subspace* of $X$.

**Lemma 5.** *If $\mathcal{B}$ is a basis for the topology of $X$ then the collection $\mathcal{B}_Y = \{B \cap Y | B \in \mathcal{B}\}$ is a basis for the subspace topology on $Y$.*

**Lemma 6.** *Let $Y$ be a subspace of $X$. If $U$ is open in $Y$ and $Y$ is open in $X$, then $U$ is open in $X$.*

**Theorem 7.** *If $A$ is a subspace of $X$ and $B$ is a subspace of $Y$, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.*

**Definition 8.** Let $X$ and $Y$ be topological spaces; let $p : X \to Y$ be a surjective map. The map $p$ is said to be a *quotient map* provided a subset $U$ of $Y$ is open in $Y$ if and only if $p^{-1}(U)$ is open in $X$.

**Definition 9.** A subset $C$ of a topological space $X$ is *saturated* with respect to the surjective map $p : X \to Y$ if $C$ contains every set $p^{-1}(\{y\})$ that it intersects.

**Definition 10.** A map $f : X \to Y$ is said to be an *open (closed) map* if for each open (closed) set $U$ of $X$, the set $f(U)$ open (closed) in $Y$.

**Definition 11.** If $X$ is a space and $A$ is a set and if $p : X \to A$ is a surjective map, then there exists exactly one topology $\mathcal{T}$ on $A$ relative to which $p$ is a quotient map. $\mathcal{T}$ is called the *quotient topology* induced by $p$.

**Definition 12.** Let $X$ be a topological space, and let $X^*$ be a partition of $X$ into disjoint subsets whose union is $X$. Let $p : X \to X^*$ be the surjective map that carries each point of $X$ to the element of $X^*$ containing it. In the quotient topology induced by $p$, the space $X^*$ is called a *quotient space* of $X$.

**Theorem 8.** *Let $p : X \to Y$ be a quotient map; let $A$ be a subspace of $X$ that is saturated with respect to $p$; let $q : A \to p(A)$ be the map obtained by restricting $p$.*

1. *If $A$ is either open or closed in $X$, then $q$ is a quotient map.*

2. *If $p$ is either an open map or closed map, then $q$ is a quotient map.*

**Theorem 9.** *Let $p : X \to Y$ be a quotient map. Let $Z$ be a space and let $g : X \to Z$ be a map that is constant on each set $p^{-1}(\{y\})$, for $y \in Y$. Then $g$ induces a map $f : Y \to Z$ such that $f \circ p = g$. The induced map $f$ is continuous if and only if $g$ is continuous; $f$ is a quotient map if and only if $g$ is a quotient map.*

**Corollary 10.** *Let $g : X \to Z$ be a surjective continuous map. Let $X^*$ be the following collection of subsets of $X$:*

$$X^* = \{g^{-1}(\{z\}) \,|\, z \in Z\}$$

*Give $X^*$ the quotient topology.*

1. *The map $g$ induces a bijective continuous map $f : X^* \to Z$, which is a homeomorphism if and only if $g$ is a quotient map.*
2. *If $Z$ is Hausdorff, so is $X^*$.*

# 3 Closure, Interior, and Limit Points

**Definition 13.** A subset $A$ of a topological space $X$ is *closed* if the set $X - A$ is open.

**Theorem 11.** *Let $X$ be a topological space. Then the following hold:*

- *$\emptyset$ and $X$ are closed.*
- *Arbitrary intersections of closed sets are closed.*
- *Finite unions of closed sets are closed.*

**Theorem 12.** *Let $Y$ be a subspace of $X$. The set $A \subset Y$ is closed in $Y$ iff it equals the intersection of a closed set of $X$ with $Y$.*

**Theorem 13.** *Let $Y$ be a subspace of $X$. If $A \subset Y$ is closed in $Y$ and $Y$ is closed in $X$, then $A$ is closed in $X$.*

**Definition 14.** Let $A$ be a subset of a topological space $X$. The *interior* of $A$, denoted Int$A$, is the union of all open sets contained in $A$. The *closure* of $A$ is the intersection of all closed sets containing $A$.

**Proposition 14.** *Let $A$ be a subspace of a topological space $X$. Then Int $A \subset A \subset \bar{A}$. If $A$ is open, then $A =$IntA; if $A$ is closed, then $A = \bar{A}$.*

**Theorem 15.** *Let $Y$ be a subspace of $X$; let $A$ be a subset of $Y$; let $\bar{A}$ be the closure of $A$ in $X$. Then the closure of $A$ in $Y$ is $\bar{A} \cap Y$.*

**Theorem 16.** *Let $A$ be a subset of the topological space $X$. The following are true:*

- *$x \in \bar{A}$ if and only if every open set $U$ containing $x$ intersects $A$.*
- *Supposing the topology of $X$ is given by a basis, then $x \in \bar{A}$ if and only if every basis element $B$ containing $x$ intersects $A$.*

**Definition 15.** If $A$ is a subset of a topological space $X$, we call $x \in X$ a *limit point* of $A$ if every neighborhood of $x$ intersects $A$ at some point other than itself.

**Theorem 17.** *Let $A$ be a subset of the topological space $X$; let $A'$ be the set of all limit points of $A$. Then we have*

$$\bar{A} = A \cup A'$$

.

**Corollary 18.** *A subset of a topological space is closed if and only if it contains all of its limit points.*

# 4 Hausdorff Spaces

**Definition 16.** A topological space $X$ is called *Hausdorff* if for each pair $x_1, x_2$ distinct in $X$, there exists neighborhoods $U_1$ and $U_2$ of $x_1$ and $x_2$, respectively, such that $U_1$ and $U_2$ are disjoint.

**Theorem 19.** *Every finite point set in a Hausdorff space is closed. This condition is called the $T_1$ axiom.*

**Theorem 20.** *Let $X$ be a topological space satisfying the $T_1$ axiom; let $A$ be a subset of $X$. Then the point $x$ is a limit point of $A$ if and only if every neighborhood of $x$ contains infinitely many points of $A$.*

**Definition 17.** A sequence $(x_n)$ of an arbitrary topological space $X$ *converges* to the point $x \in X$ if for any neighborhood $U$ of $x$, there exists a positive integer $N$ such that $x_n \in U$ if $n \geq N$.

**Theorem 21.** *If $X$ is a Hausdorff space, then a sequence of points of $X$ converges to at most one point of $X$.*

# 5 Continuous Functions

**Definition 18.** Let $X, Y$ be topological spaces. The map $f : X \to Y$ is said to be *continuous* if for every open set $V$ in $Y$, $f^{-1}(V)$ is open in X.

**Proposition 22.** *If $\mathcal{B}$ is a basis for the topology on $Y$, $f$ is continuous if and only if $f^{-1}(B)$ is open in $X$ for all $B \in \mathcal{B}$*

**Example 1.** For instance, $f : \mathbb{R} \to \mathbb{R}$ with the standard topology where $f(x) = x$ is continuous; however, $f : \mathbb{R} \to \mathbb{R}_l$ with the standard topology where $f(x) = x$ is not continuous.

**Theorem 23.** *Let $X, Y$ be topological spaces with $f : X \to Y$*
*The following are equivalent:*

- *$f$ is continuous.*
- *For all closed $B \subseteq Y$, $f^{-1}(B)$ is closed.*
- *For all $x \in X$ and each neighborhood $V$ of $f(x)$, there exists a neighborhood $U$ of $x$ such that $f(U) \subseteq V$*
- *For all $A \subseteq X, f(\bar{A}) \subseteq f(\bar{A})$.*

**Theorem 24.** *There are various ways of constructing toplogical spaces.*
*Let $X, Y, Z$ be topological spaces. Here are a few:*

- *$f : X \to Y$ with $f(x) = y_o$ for some fixed $y_o \in Y$ is continuous.*
- *If $A \subseteq X$, then $f : A \to X, f(a) = a$ for $a \in A$ is continuous*

- *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then so is $g \circ f : X \to Z$*

**Theorem 25** (Pasting Lemma)**.** *Let $X$ be a topological space such that $X = A \cup B$ with $A, B$ closed in $X$. Let the functions $f : A \to Y, g : B \to Y$ be continuous. If $f(x) = g(x)$ for all $A \cap B$, we obtain a well defined function $h : X \to Y$ where $h(x) = f(x)$ for $x \in A$ and $h(x) = g(x)$ for $x \in B$.*

# 6 Homeomorphisms

**Definition 19.** Let $X, Y$ b topological spaces. A bijection $f : X \to Y$ is called a *homeomorphism* if both $f$ and $f^{-1}$ are continuous.

**Definition 20.** Suppose $f : X \to Y$ be both injective and continuous, with $X, Y$ being topological spaces. Let $Z = f(X) \subseteq Y$ with the subspace topology. If $f' : X \to Z$ is a homeomorphism, then $f : X \to Y$ is called a topological *embedding*.

**Example 2.** Any open interval $(a, b)$ is homeomorphic to $\mathbb{R}$. For instance, consider $f : (-1, 1) \to \mathbb{R}$ with $f(x) = x/1 - x^2$. This is a homeomorphism.

**Example 3.** Not all continuous bijective functions are homeomorphisms. For instance, Consider $f : [0, 2\pi) \to \mathbb{S}^1$ with $f(x) = (cos(x), sin(x))$. This function is continuous and bijective, yet $[0, 2\pi)$ fails to be homeomorphic to the circle $\mathbb{S}^1$.

# 7 Connectedness

**Definition 21.** Let $X$ be a topological space. A *separation* of X is a pair of disjoint, open, nonempty subsets $U, V$ of $X$ such that $U \cup V = X$.

**Definition 22.** $X$ is called *disconnected* if it has a separation. If $X$ has no separations, it is called *connected*.

**Example 4.** The discrete topology is clearly disconnected as long as it contains at least two elements.
$\mathbb{R}$ with the standard topology is connected.

**Theorem 26.** *The image of a connected space under a continuous map is connected.*

**Corollary 27.** *If $X, Y$ are homeomorphic topological spaces, then $X$ is connected if and only if $Y$ is connected.*

**Theorem 28.** *Let $X$ be a connected space with $f : X \to \mathbb{R}$ continuous.
If $p, q \in f(x)$ and $p \leq r \leq q$ for some $r \in \mathbb{R}$ then $r \in f(x)$.*

**Corollary 29** (Intermediate value theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous such that $s \in \mathbb{R}$ lies between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ such that $f(c) = s$.*

**Corollary 30** (Brouwer's Fixed Point Theorem)**.** *Any continuous function $f : [-1, 1] \to [-1, 1]$ has a fixed point. So there is some $c \in [-1, 1]$ such that $f(c) = c$.*

**Definition 23.** Given $x$ and $y$ in a topological space X, a *path* in $X$ from $x$ to $y$ is a continuous function $f : [a, b] \to X$ such that $f(a) = x, f(b) = y$.
A topological space $X$ is said to be *path connected* if every pair of points can be joined by a path in $X$.

**Definition 24.** Let $X$ be a connected topological space. A *cutset* of $X$ is a subset $S \subseteq X$ such that $X - S$ is disconnected.
A *cutpoint* of $X$ is a singleton set cutset.

# 8 Compactness

**Definition 25.** A collection $\mathcal{A}$ of subsets of a space $X$ is said to *cover*$X$ if $X = \cup_{A \in \mathcal{A}} A$
$\mathcal{A}$ is an *open cover* if each of the subsets is open in $X$.

**Definition 26.** $X$ is said to be *compact* if every open cover has a finite subcover, that is, a finite subcollection that also covers $X$.

**Theorem 31.** *The image of a compact space under a continuous map is compact.*

**Theorem 32.** *Every compact subspace of a Hausdorff space is closed.*

**Theorem 33.** *Every closed subspace of a compact space is compact.*

**Theorem 34.** *Let $f : X \to Y$ be bijective and continuous. If $X$ is compact and $Y$ is hausdorff, then $f$ is a homeomorphism.*

**Theorem 35.** *Any closed interval $[a, b]$ of $\mathbb{R}(a, b \in \mathbb{R})$ is compact.*

**Definition 27.** A set $A$ in $\mathbb{R}^n$ is said to be *bounded* if there is an $m > 0$ such that the distance between any $x, y \in A$ is less than or equal to $m$.

**Theorem 36.** *$A \subseteq \mathbb{R}^n$ is compact if and only if it is closed as well as bounded.*

**Theorem 37.** *Let $f : X \to \mathbb{R}$ be continuous and $X$ be compact. Then there exist $c, d \in X$ such that $f(x) \subseteq [f(c), f(d)]$ for all $x \in X$.*

# 9 Metric Spaces

**Definition 28.** A metric on a set $X$ is a function $d : X \times X \to \mathbb{R}$ such that

- $d(x, y) \geq 0$ for all $x, y \in X$
- $d(x, y) = 0$ if and only if $x = y$
- $d(x, y) = d(y, x)$ for all $x, y \in X$
- $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$

The pair $(X, d)$ is called a metric space.

**Fun Fact 1.** $d(x, y) \geq 0$ is implied by the other conditions, but it is often included in the definition anyway as it is so fundamental (and not immediately obvious otherwise).

**Example 5.** The traditional notion of distance in Euclidean space is a metric on $\mathbb{R}^n$, as is the "Taxicab" metric.

**Example 6.** As a more discrete (and discreet) example, $V^n = \{0, 1\}^n$, the set of binary strings of length $n$, has as a metric the Hamming distance, which counts the number of places (among the n of them) where two strings differ.

**Fun Fact 2.** A metric on a set $X$ induces a metric on a subset of $X$.

**Example 7.** The Hamming distance on $V^n$ is equivalent to the metric inherited by $V^n$ as a subset of $\mathbb{R}^n$ with the taxicab metric.

**Definition 29.** Let $(X, d)$ be a metric space. For $x \in X$, $\epsilon > 0$,
$B_d(x, \epsilon) = \{y \in X | d(x, y) < \epsilon\}$ is called the open ball of radius $\epsilon$ centered at $x$.

**Theorem 38.** *$\mathcal{B} = \{B_d(x, \epsilon) | x \in X, \epsilon > 0\}$ is a basis for a topology on $X$. The topology it generates, "the topology induced by d", is called a "metric topology".*

**Definition 30.** A topological space $X$ is said to be metrizable if there exists a metric that induces the topology on $X$.

**Fun Fact 3.** In the pursuit of topology, it is often more important to know that a space is metrizable than to know anything about specific metrics on it, as metrizable spaces are nice.

**Proposition 39.** *Any metrizable space is Hausdorff.*

**Proposition 40.** *If topological spaces $X$ and $Y$ are metrizable, then the $\epsilon - \delta$ definition of continuity of functions between them from calculus/analysis is equivalent to the topological definition of continuity.*

**Proposition 41.** *If $X$ is metrizable, $A \subseteq X$, some sequence in $A$ converges to $x$ for all $x \in \bar{A}$. (The converse is true for all topological spaces.)*

**Proposition 42.** *If $X$ is metrizable, $f : X \to Y$ is continuous if all convergent sequences in $X$ are preserved by $f$. (The converse is true for all topological spaces.)*

**Definition 31.** For metric spaces $(X, d_X)$ and $(Y, d_Y)$, an isometry (or isometric embedding) is a function $f : X \to Y$ such that for all $x, y \in X$, $d_X(x, y) = d_Y(f(x), f(y))$. Isometries can be thought of as structure preserving maps between metric spaces.

**Theorem 43.** *(Urysohn metrization) If $X$ is a $T_1$, $T_3$, second countable topological space, then $X$ is metrizable.*

**Theorem 44.** *(Tietze extension) Let $X$ be metrizable, and $A \subseteq X$ closed.*

- *Any continuous $f : A \to \mathbb{R}$ can be extended to a continuous $F : X \to \mathbb{R}$*
- *Any continuous $f : A \to [a, b]$ can be extended to a continuous $F : X \to [a, b]$*

# 10 Dynamical Systems

**Definition 32.** Let $X$ be a topological space. A discrete time dynamical system is $(X, f)$ where $f : X \to X$ is continuous. ($X$ need only be a nonempty set in general, but working with a topology and a continuous function makes it more exciting.)

**Fun Fact 4.** There are also continuous time dynamical systems, which often take the form of differential equations.

**Definition 33.** Let $(X, f)$ be a dynamical system, $x \in X$.

- The orbit of $x$ under $f$ is the sequence $\{f^n(x) | n \in \mathbb{N} \text{ (or sometimes } \mathbb{Z})\}$
- $x$ is a fixed point under $f$ if $f(x) = x$
- $x$ is a fixed point of period $m \in \mathbb{Z}^+$ if $f^m(x) = 0$, and $m$ is the lowest such integer

**Definition 34.** For dynamical system $(X, f)$ with fixed point $X^*$

- $x^*$ is stable if for all open neighborhoods $U \ni x^*$, there exists an open neighborhood $V \ni x^*$ such that for all $x \in V$, the orbit of $x$ is contained in $U$
- $x^*$ is asymptotically stble if $x^*$ is stable and there is an open neighborhood $U \ni x^*$ such that for all $x \in U$, $f^n(x)$ converges to $x^*$.
- $x^*$ is neutrally stable if it is stable but not asymptotically stable
- $x^*$ is unstable if it is not stable

**Example 8.** An example of of a continuous dynamical system is a marble rolling in a bowl, where the space is the surface of the bowl and $f^n(x)$ is thought of as the position at time $n$ of a marble released from position $x$ on the surface of the bowl at time 0. Clearly the bottom-most point of the bowl is a fixed point. If friction is considered, then the marble released anywhere on the bowl will eventually 'converge' to the fixed point at the bottom of the bowl, which is then asymptotically stable. However, without friction, the marble can roll back and forth across the bottom of the bowl indefinitely, so the bottom is then neutrally stable. If the bowl is turned upside down and the marble rolled on top of it, the top-most point of the bowl is an unstable fixed point.

**Definition 35.** Dynamical systems $(X, f)$ and $(Y, g)$ are called semi-conjugate if there is a surjective continuous function $h : X \to Y$ such that $h \circ f = g \circ h$. If $h$ is a homeomorphism, they are called topologically conjugate.

**Definition 36.** A subset $A$ of a topological space $X$ is said to be dense if $\bar(A) = X$.

**Definition 37.** A dynamical system $(X, f)$ is said to be chaotic (or have chaos) if

- The set of periodic points is dense in $X$
- For all open $U, V \subseteq X$, there exists $x \in U$, $n \in \mathbb{Z}^+$ such that $f^n(x) \in V$

**Example 9.** The tent map, which at successive iterations looks like increasingly many increasingly thin tents when graphed in the plane, it chaotic. This makes sense, as if your tent becomes increasingly thin, it becomes increasingly likely that the height of the tent above where you are sitting in it is within some open range of possible heights.

**Proposition 45.** *Topological conjugacy preserves chaos. This makes a topological conjugacy in a sense a structure preserving map between dynamical systems.*

**Definition 38.** Let $(X, d)$ be a metric space. A continuous function $f : X \to X$ has sensitive dependence on initial conditions if there exists a $\delta > 0$ such that for all $x \in X$, $\epsilon > 0$, there exists $y \in B_d(x, \epsilon)$, $n \in \mathbb{Z}^+$ such that $d(f^n(x), f^n(y)) > \delta$. In other words, no matter how close two points may be, they always grow apart after some number of iterations of $f$.

**Theorem 46.** *If $X$ is an infinite metric space, and $f : X \to X$ is continuous and chaotic, then $f$ has sensitive dependence on initial conditions.*

**Fun Fact 5.** This means that in such a space, no approximation of $x$ can accurately be used to predict $f^n(x)$ for large $n$, as any point, even those very close to $x$, will not remain close to $x$ after many iterations of $f$. This notion makes chaotic systems more intuitively chaotic. It also makes the lives of scientists more chaotic.

# 11 Homotopy

**Definition 39.** An invariant of topological spaces is something space that is preserved under homeomorphism. A complete invariant is an invariant that determines a space up to homeomorphism, but those are hard to find.

**Definition 40.** A functor is a function from one class of objects to another that also acts on the morphisms between the objects (such as the functions between topological spaces). Functors are particularly good for describing relationships between different classes of objects, as they preserve the existing relationships between objects within a class.

**Definition 41.** A homotopy between two continuous functions $f, g : X \to Y$ is a continuous function $F : X \times [0, 1] \to Y$ such that for all $x \in X$, $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. If such a homotopy exists, $f$ and $g$ are said to be homotopic.

**Definition 42.** A loop with base point $q \in X$ is a path from $q$ to $q$, or equivalently an embedding of $S^1$ in $X$.

**Definition 43.** A path homotopy between paths $f$ and $g$ from $x$ to $y$ in $X$ is a homotopy between them such that each intermediate function ($F(x, t)$ for some fixed $t$) is also a path from $x$ to $y$.

**Definition 44.** The constant path $C_q$ is a path in $X$ that is a constant function to $q$. A loop $f$ is called null-homotopic if it is path homotopic to $C_{f(0)}$

**Definition 45.** For fixed $q \in X$, path homotopy is an equivalence relation on the set of loops with base point $q$. The fundamental group of $X$, $\pi_1(X,q)$ is the set of path homotopy classes of loops with base point $q$ under the group operation of path multiplication (which is essentially the concatenation of two paths). The identity class is that containing $C_q$.

**Fun Fact 6.** If $X$ is path connected, then $\pi_1(X,q)$ is independent of $q$ up to isomorphism and written just as $\pi_1(X)$.

**Definition 46.** $X$ is called simply connected if it is path connected and $\pi_1(X)$ is trivial.

**Example 10.** $\pi_1(S^1) = \mathbb{Z}$. $\pi_1(S^2) = 0$.

**Fun Fact 7.** If $X$ and $Y$ are path connected, $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$. Therefore $\pi_1(T^2) = \mathbb{Z}\times\mathbb{Z}$. As this is a homeomorphism invariant, $S^2$ and $T^2$ clearly not homeomorphic.

**Definition 47.** For continuous function $\phi : X \to Y$, define, for any $q \in X$, $\phi_* : \pi_1(X,q) \to \pi_1(Y,\phi(q)) : [f] \mapsto [\phi \circ f]$. $\phi_*$ is a well defined group homomorphism (multiplication preserving function between groups).

**Fun Fact 8.** $\pi_1$ and the corresponding operation on functions defined above satisfy the ‘functor laws’, making this correspondence between groups and topological spaces a functor.

**Corollary 47.** *If $\phi : X \to Y$ is a homeomorphism, then $\phi_* : \pi_1(X) \to \pi_1(Y)$ is an isomorphism (bijective homomorphism) of groups.*

**Fun Fact 9.** There are also higher homotopy groups $\pi_n$ where the loops are replaced with embedding of higher dimensional spheres in the space. In most cases, consistent with intuition, $\pi_n(S^m)$ is trivial when $n \neq m$, as low dimensional spheres can shrink to points in high dimensional spheres, and high dimensional spheres cannot wrap themselves continuously around lower dimensional spheres to get nontrivial higher homotopy classes. However, $\pi_3(S^2) = \mathbb{Z}$, due to the existance of something wonderful called the Hopf bundle, which projects $S^3$ onto $S^2$ surjectively, generating nontrivial higher homotopy classes.

# 12 Fixed Point Theorems

**Definition 48.** The n-dimensional disk is defined as $\mathbb{D} = \{(x_1, ..., x_n) \in \mathbb{R}^n | \sum_i x_i^2 \leq 1\}$

**Theorem 48.** *(Brouwer's) Any continuous function $f : \mathbb{D} \to \mathbb{D}$ has a fixed point*

**Definition 49.** A set valued function $f : X \to_S Y$ is a function from $X$ to the power set of $Y$, so for $x \in X$, $f(x) \subseteq Y$. A fixed point of a set valued function $f : X \to_S X$ is a point $x^* \in X$ such that $x^* \in f(x^*)$.

**Definition 50.** A subset $A \subseteq \mathbb{R}^n$ is called convex if for all pairs of points in $A$, the straight line segment between them is also in $A$.

**Fun Fact 10.** A set valued function from $X$ to $Y$ can be graphed in $X \times Y$ by, at each point $x \in X$, shading all of the points $\{x \times y | y \in f(x)\}$. The graph of the function is then most likely region in the cartesian product of $X$ and $Y$ unlike the more traditional paths graphed for the usual functions. One major aesthetic difference is that the ‘vertical line test’ does not work for these graphs.

**Theorem 49.** *(Kakutani) If $X$ is a convex, compact subset of $\mathbb{R}^n$, the graph of $f$ is closed in $X \times X$, and for all $x \in X$, $f(x)$ is nonempty and convex, $f : X \to_S X$ has a fixed point.*

**Definition 51.** A polyhedron in $\mathbb{R}^n$ is a bounded subset of $\mathbb{R}^n$ that can be expressed as an intersection of half-spaces. Polyhedra are convex and compact in $\mathbb{R}^n$.

# 13 Game Theory

**Definition 52.** Say there are some number of players, each of whom has some number of choices of moves to make in a turn. Each player receives a payoff determined by their chosen move for the turn and the moves of the other players, and knows what payoffs each player receives for each combination of moves. The goal is to get the most payoff.

A strategy that is deterministic, or always the same given the same parameters, is unwise, as then other players can tailor their strategies taking the determined choice into account. Therefore, mixed strategies, which assign a probability value to each possible move, are used to avoid providing opponents with too much information.

Given the strategies used by the other players, one can determine the set of strategies that optimize the expected payoff (strategies which give high probabilities only to moves that are likely to have a high payoff). A collection of strategies, one for each player, is said to solve the game if each strategy is optimal given the others. That is, every player is using the best possible strategy based on the strategies of the other players. Such a collection of strategies is called a Nash equilibrium.

**Theorem 50.** *(Nash's Theorem) Every n-person game has a Nash Equilibrium*

**Fun Fact 11.** While this theorem is interesting for its own sake, as it means that every game (like monopoly) has strategies that are simultaneously optimal for all players, it is particularly exciting that the proof of this theorem is topological. It relies on the shape of the space of optimal solutions, and on (a choice of) the fixed point theorems discussed above.

# 14 Manifolds

**Definition 53.** An n-manifold ($n \geq 1$) $X$ is a topological space that is Hausdorff, second countable, and locally Euclidean, meaning that each point has a neighborhood homeomorphic to $\mathbb{R}^n$ (or equivalently an open subset of $\mathbb{R}^n$).

**Example 11.** Nice spaces like spheres and tori are manifolds, as they are Hausdorff, have countable bases, and most importantly, small neighborhoods are clearly homeomorphic to subsets of Eulidean space. Spaces that intersect themselves or have just a point of contact between two higher dimensional components are generally not manifolds, such as the figure 8 in $\mathbb{R}^2$, which has one problematic point where the lines cross.

**Theorem 51.** *Up to homeomorphism, the only connected and compact 1-manifold is $S^1$, and the only connected non-compact 1-manfiold is $\mathbb{R}$. All other $1-manifolds$ are countable disjoint unions of copies of $S^1$ and $\mathbb{R}^{`}$.*

**Definition 54.** A 2-manifold is called non-orientable if it has an embedded mobius band, and otherwise it is orientable.

**Example 12.** The Klein bottle is non-orientable as it is simply a mobius band witth a disk glued on along the boundary. The sphere and torus are orientable as there is a consistent notion of moving out of the surface.

**Theorem 52.** *Every compact, connected, oriented 2-manifold is homeomorphic to $S^2$ or some connected sum of copies of T. Every non-oriented 2-manifold is homeomorphic to some connected sum of copies of $RP(2)$*

**Fun Fact 12.** Every compact connected 3-manifold can be found by gluing together solid tori along different knots.

# 15 Knots

**Definition 55.** A knot $K$ is a smooth embedding $K : S^1 \hookrightarrow S^3$. This means that all knots are homeomorphic to $S^1$ and therefore to each other. Knots are projected into two-dimensional pictures that visually represent when arcs cross over each other in 3-space.

**Definition 56.** Two knots are called isotopic if they have a homotopy between them that is still an embedding at each intermediate stage. Such a homotopy is called an isotopy.

**Definition 57.** The crossing number of a knot is the minimum number of crossings in any projection of the knot.

**Fun Fact 13.** If the embeddings were not required to be smooth, there could be a knot with infinite crossing number.

**Definition 58.** The genus of a knot $K$ is the minimum genus of any orientable surface with boundary $K$.

**Theorem 53.** *(Seifert's) Every knot in $S^3$ bounds an orientable surface.*

**Fun Fact 14.** The surface bounded by a knot can be constructed by the algorithm in the proof of this theorem, however it may not be the surface with the lowest possible genus.

**Definition 59.** Two knots are concordant if they can be embedded in $\mathbb{R}^3 \times \{0\}$ and $\mathbb{R}^3 \times \{1\}$ respectively with a continuous cylinder between them in $\mathbb{R}^3 \times [0, 1]$.

**Proposition 54.** *Knot concordance is an equivalence relation, and the concordance classes of knots form a group under the connected sum operation.*

**Fun Fact 15.** This notion of loosening the notion of equality to get a structure (here algebraic) that is more easily understood is incredibly common in algebraic topology and mathematics at large.