Simple Homotopy Theory and K_1

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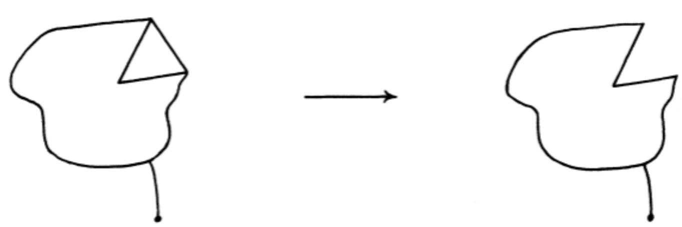
Homotopy theory begins with the intuitive idea that the most important information about a space should be independent of continuous deformations. In this spirit, maps can be seen as morally the same if one can be deformed into the other by a homotopy, and similarly spaces with opposing maps between them where both composites are deformations of the identity are equivalent by all meaningful accounts.

One could hope that out of this enlightened perspective comes a beautiful theory of simple objects that perfectly capture the homotopy types of spaces, and between any two homotopy equivalent spaces the deformation between them is easy to see. But alas, homotopy theory is hard and sometimes equivalences come without a clear picture of what the deformation looks like. If only there were a way to decompose each homotopy equivalence into nice little geometric steps...

Well there isn't, but J. H. C. Whitehead tried very hard to do so and in the process uncovered an elegant way to measure how geometric (in a sense) the equivalences are in a particular homotopy type, using K-theory!

1. Simple Homotopy

Whitehead first used simplicial complexes to describe simple geometric steps for deforming from one space to another. An n-horn in a complex is an arrangement of $(n-1)$-simplices resembling an n-simplex with the interior and one face removed, written $\wedge^n$. In this setting, the fundamental deformation will be finding an n-horn in a complex L and adding in a new n-simplex filling it in with a new $(n-1)$-simplex as the remaining face to form a complex X.



Definition 1. For A a subcomplex of X, there is an *elementary collapse* from X to A if

$X = A \cup_{\wedge^n} \Delta^n$ for some n-cell of X. In other words, X is obtained from A by adding one new n-cell along a horn in A and one new $(n-1)$-cell as its last remaining face.

This definition of elementary collapse works just as well for CW complexes X, where each n-cell is determined by a map of spaces $(\Delta^n, \partial\Delta^n) \to (X^{(n)}, X^{(n-1)})$ (if one thinks of the n-ball as a simplex). We will assume going forward that (X, A) is a finite CW pair, and also that all maps of CW complexes are cellular.

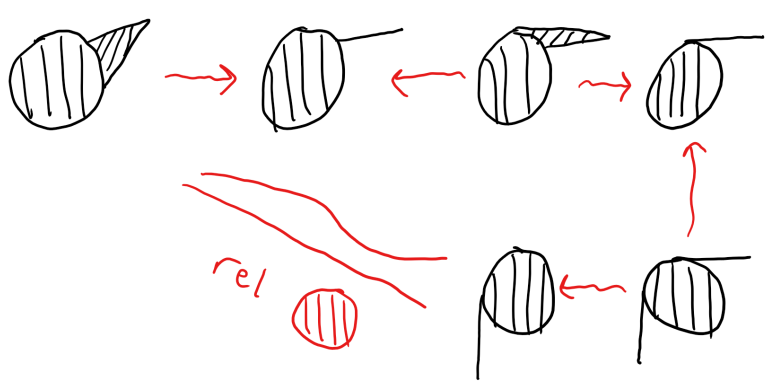
If X collapses to A in this sense, it is in fact by a deformation retraction, specifically the one acting as the identity on A and your favorite deformation retraction $\Delta^n \to \wedge^n$. These elementary collapses are nice little geometric homotopy equivalences, and life would be simple indeed if any homotopy equivalence could be built from them.

Definition 2. X and Y have the same *simple homotopy type* if they are related by a sequence $X = X_0, X_1, ..., X_k = Y$ where for each $0 \le i < k$ there is an elementary collapse from either X_i to X_{i+1} or vice versa.

Definition 3. For such X and Y, a *simple homotopy equivalence* $f : X \to Y$ is a map homotopic to the composite $X = X_0 \to X_1 \to \cdots \to X_k = Y$ of inclusions and retractions.

Definition 4. If (X_i, Z) in the above is a CW pair for all i and fixed Z with each map $X_i \to X_{i+1}$ the identity on Z, then f is a simple homotopy equivalence relative to Z and we write $X \simeq_s Y$ rel Z.

Example 5.



Simple homotopy equivalences are aptly named! For any old homotopy equivalence, the quasi-inverse could appear completely unrelated, with the accompanying homotopies full of dizzying twists and turns. Or worse, the inverse could be known to exist by model category axioms without having ever been photographed. A simple homotopy equivalence is merely the result of adding and removing simplices at the edges of a space.

In a simpler world, one might expect all equivalences to be of this form, or at least simple homotopy types to be the same as homotopy types. They're not, but we can study the simple homotopy types within a single homotopy type using K-theory.

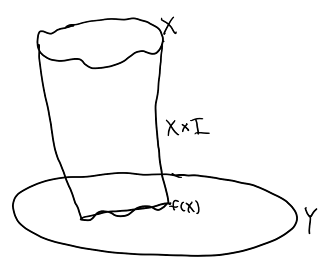
2. Geometric Whitehead Group

We can start comparing simple homotopy types to homotopy types by fixing A and looking at the simple homotopy classes of spaces that deformation retract to A.

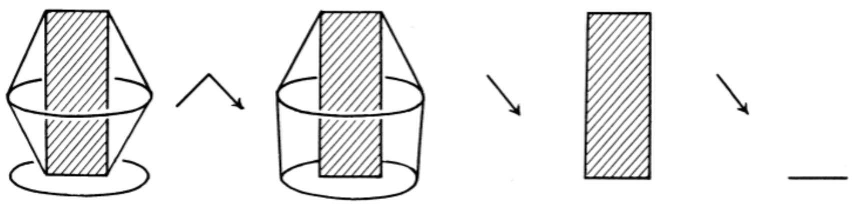
Definition 6. Define $Wh(A)$ to be the set of CW pairs (X, A) with a deformation retraction $X \to A$, modulo the equivalence relation of simple homotopy equivalence rel A. Write $[X, A]$ for the equivalence class of (X, A) in $Wh(A)$.

$Wh(A)$ will be called the *Whitehead group* of A with the operation $[X, A] + [Y, A] = [X \cup_A Y, A]$. It is not hard to show this operation is well defined, and we immediately see it is associative and commutative with $[A, A]$ as an identity by the same properties of $\cup$.

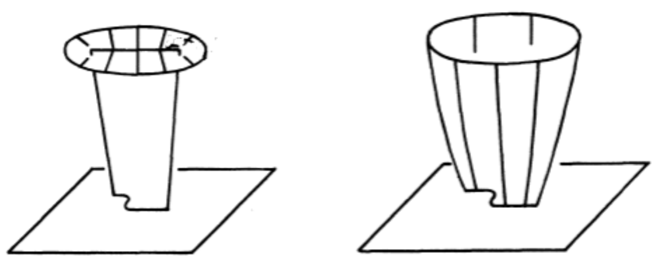
Recall that the mapping cylinder of a map $X \to Y$ is the space $M_f = X \times I \sqcup Y/(x, 1) \sim f(x)$, with the inclusions $X \hookrightarrow M_f : x \mapsto (x, 0)$ and $Y \hookrightarrow M_f : y \mapsto y$.



With $D : X \to A$ a choice of deformation retraction, define $[X, A]^{-1} = [M_D \cup_X M_D, A]$ with A included as above into one of the copies of M_D (see picture below). Each step in the following picture can be demonstrated as a simple homotopy equivalence rel A, together proving that $[(M_D \cup_X M_D) \cup_A X, A] = [A, A]$.



In this demonstration X is the disk, A is a line segment included into the center of the disk, and the components of the union written left to right appear in the picture from top to bottom. For $f : A \to B$, we can define $f_*[X, A]$ as either $[X \cup_A M_f, B]$ or $[X \cup_f B, B] \in Wh(B)$. Using the same example pictured above, if f is a map from the line segment A into the square B, then $f_*[X, A]$ can accordingly be pictured as either of the following:



With this picture in mind, f_* is rather clearly a group homomorphism $Wh(A) \to Wh(B)$. One can verify that this gives a functor Wh from CW complexes to abelian groups, which furthermore extends to a functor from the homotopy category of CW complexes as if $f \simeq g$ then $f_* = g_*$.

The Whitehead group of A describes the simple homotopy types of deformation retractions to A, but we were originally interested in characterizing homotopy equivalences in general, which $Wh(A)$ provides some vocabulary for:

Definition 7. For $f : A \to B$ a homotopy equivalence, the Whitehead torsion $\tau(f) = f_*[M_f, B] = [M_f \cup_A M_f, B] \in Wh(B)$.

Note that there is a homotopy equivalence to A with torsion any fixed element of $Wh(A)$, as for $D : X \to A$ a deformation retraction for (X, A) as above, $\tau(D) = [X, A]^{-1} \in Wh(A)$.

The following proposition gives a taste of what homotopical information goes into $Wh(A)$, as we will elaborate on using K-theory in the following sections.

Proposition 8. *If A is a simply connected CW complex then $Wh(A) = 0$. In other words, any deformation retraction to a simply connected space is a simple homotopy equivalence.*

3. Algebraic Whitehead Group

There is a K-theoretic construction of something else called the Whitehead group, which will ultimately relate back to simple homotopy theory. Here we assume all rings R are "nice" in the sense that any finitely generated free module over R has a well defined dimension, which will be true for the examples we care about.

Recall that $GL(n, R)$ is the group of invertible n by n matrices over R and we have inclusions $GL(n, R) \hookrightarrow GL(n + 1, R)$ by $M \to \begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix}$.

Definition 9. The group $GL(R) = colim_n(GL(n, R))$ is the colimit of these inclusions, consisting of infinite dimensional matrices over R that in high enough dimension look like the identity.

We will be interested in the abelianization of $GL(R)$, which will ultimately be defined as $K_1(R)$. For G a subgroup of the units $R^\times$, let E_G be the group generated by the commutator subgroup of $GL(R)$ and diagonal matrices with only entries from G.

Definition 10. $K_G(R)$ is the quotient $GL(R)/E_G$.

$K_G(R)$ is a quotient of $GL(R)^{ab}$, so E_G is normal and $K_G(R)$ is abelian. Amusingly, this notation includes K_1 in a setting where K_n doesn't even make sense. This is all we need to define something else called the Whitehead group:

Definition 11. For G a group, define the Whitehead group $Wh(G)$ as $K_T(R)$, where $R = \mathbb{Z}[G]$ and $T = G \sqcup (-G)$.

We write τ for the quotient map $GL(R) \to K_G(R)$, also called the torsion. Note that K_G extends to a functor from pairs (R, G) to abelian groups in a straightforward way, Wh to a functor from groups to abelian groups.

4. Connections and Answers

We now have two functors to abelian groups both suggestively called $Wh(-)$, one taking CW complexes and the other groups. Naturally, this naming convention is well motivated by the following:

Theorem 12. *There is a natural isomorphism of functors from CW complexes to abelian groups $Wh(A) \to \oplus_i Wh(\pi_1 A_i)$ where A_i are the connected components of A.*

We won't prove this here, but the homomorphism is defined on $[X, A]$ (for connected A) by constructing the $\mathbb{Z}[\pi_1 A]$-chain complex $C(\tilde{X}, \tilde{A})$ from the universal covers of X, A and putting it into a special form with two nontrivial components and an invertible differential between them. That map of $\mathbb{Z}[\pi_1 A]$-modules can be expressed as a matrix and its torsion gives an element of $Wh(\pi_1 A)$.

This isomorphism allows for far easier proofs and computations about Whitehead torsion of homotopy equivalences, such as the following.

Proposition 13. *Homotopic homotopy equivalences have the same torsion.*

Proposition 14. *The torsion τ restricts to a homomorphism from the group $Eq(A) = \{A \xrightarrow{\simeq} A\}/\simeq$ to $Wh(A)$.*

Now to return to the original questions. We can use torsion to characterize when a homotopy equivalence is simple:

Theorem 15. *A homotopy equivalence is simple if and only if its torsion is 0.*

As not all Whitehead groups are trivial (some lens spaces provide counterexamples), not all homotopy equivalences are simple.

Comparing homotopy types to simple homotopy types is a bit more subtle however, as a particular homotopy equivalence not being simple doesn't mean there doesn't exist *some* simple one between the same spaces. For A to have as its simple homotopy type its entire homotopy type, $Wh(A)$ doesn't need to be trivial as there can be homotopy equivalences $A \to A$ with nonzero torsion (A is nonetheless in its own simple homotopy type).

Theorem 16. *The simple homotopy class of A is the same as the homotopy class of A iff $\tau|_{Eq(A)} : Eq(A) to Wh(A)$ is surjective.*

This characterization has led to both concrete examples and counterexamples!

- $Wh(\mathbb{Z}^n) = 0$, so any space with π_1 finitely generated free abelian has only simple homotopy equivalences.
- Any finite connected 2-complex with cyclic π_1 of cardinality greater than 6 has non-trivial Whitehead group but satisfies the condition in the theorem.
- Most lens spaces do not satisfy the condition in the theorem.

The algebraic reformulation of the Whitehead group using K-theory makes this rather intractible geometric situation far easier to reason about, and ultimately provides an answer to how geometric our homotopy theory can be.

5. References

1. Cohen, M. M. (1973). *A Course in Simple-Homotopy Theory.* New York: Springer-Verlag.