Homotopy Theory Primer for HoTT

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Homotopy Type Theory uses intuition from algebraic topology and higher category theory to justify subtle ideas in type theory, then turns around to apply powerful tools from type theory to reason about homotopy theory and categories. These notes are meant to develop some intuition for homotopy and category theoretic ideas to help one feel comfortable with topological explanations of type theory concepts, and eventually provide context for some of the applications of type theory back to algebraic topology. The main ideas I want to convey here are that:

- Paths and homotopies are essentially the same thing
- They allow spaces and categories to be reasoned about in similar ways
- Sometimes homotopy makes scary looking spaces easier to understand

1. Cats, Spaceships, and Space People

While I intend to talk a lot here about topological spaces, I don't want to spend time actually telling you what they are. Instead, I will rely on a basic familiarity with disks in various dimensions to define the 'nice' spaces that algebraic topologists actually deal with. Someone interested in how to define all of these terms topologically could consult Munkres' book, but the details of that are not particularly helpful for developing intuition about homotopy.

Definition 1. $S^n = \{\vec{x} \in \mathbb{R}^{n+1} : |x| = 1\}$, and $D^n = \{\vec{y} \in \mathbb{R}^n : |y| \leq 1\}$.

These are topological spaces! And because they are defined as subsets of Euclidean space, we can rely on what we know about that to avoid going into detail about the point-set topology.

Remark 2. The boundary of a disk is the sphere of one dimension lower, written $\partial D^n = S^{n-1}$, except $D^0 = \{*\}$

Definition 3. A CW complex X is defined as $\bigcup_n X_n$, where $X_0 = \bigsqcup_i \{v_i\}$ is a set of discrete points and $X_n = X_{n-1} \cup_{f_1} D^n \cup_{f_2} D^n ...$ where the 'attaching map' f_i is a continuous function $\partial D^n \to X_{n-1}$ and $X_{n-1} \cup_{f_1} D^n$ means the union of the two spaces with each point x in $\partial D^n \cong S^{n-1}$ identified with (or 'glued to') the point $f(x)$ in X_{n-1}.

Example 4. Build a spaceship, spheres, disks, take requests

Since we will only care about CW complexes, which are built from gluing together disks along spheres, I won't define what it means for a function to be continuous. The interested reader could guess at a reasonable definition in terms of coordinates or consult Munkres, but it is now quite honest to say a function is continuous if it doesn't break anything apart where it shouldn't. For what it's worth, most topologists (space people) rarely use a more technical approach.

I assume some basic familiarity with categories, but the following examples will come up a lot:

Example 5. Let G be a group. G can equivalently be thought of as a category with just one object and morphisms the elements of G. The identity of G behaves as the identity morphism, and composition is given by multiplication in G. As each element of a group must have an inverse, all of the morphisms in this category are isomorphisms, and one can observe that any category with a single object and only isomorphisms forms a group in this sense.

Definition 6. A groupoid is a category in which all morphisms are isomorphisms. This generalizes the notion of a group as a category described above to 'groups with multiple objects'.

Example 7. Draw some boring groupoids, maybe one for congruences of planar triangles.

We will see that groupoids with some higher dimensional structure start to resemble spaces.

2. Paths on Paths on Paths on Paths

Since the spaces we care about are built out of disks, we can think of them as locally resembling Euclidean space in most places. This makes it easy to envision what a path in a space looks like.

Definition 8. Let $I = [0, 1]$, called the unit interval.

Definition 9. For two points x and y in a space X, a path from x to y is a continuous map $\gamma : I \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

These are the traditional paths, which describe how one can 'connect' two points in a space. But I promised that higher groupoids would be similar to spaces, so everything I do for spaces will have an analogue for groupoids.

Remark 10. For a groupoid, think of the objects in the category as points and the (iso)morphisms as paths between them.

But the notion of a path is really more general than this, encompassing how to continuously move between a pair of objects of any sort in a space (like paths themselves!).

Definition 11. For spaces X and Y, their product is the set $\{(x, y) : x \in X, y \in Y\}$ with a CW structure formed by products of the cells in X and Y. When not ambiguous, write X^n for $X \times \cdots \times X$ (n times).

Example 12. $\{*\} \times X \cong X$, draw CW structure for I^2

Paths between points (which look like $\{*\}$ are described by maps $I = \{*\} \times I \to X$. This gives us an idea of how to describe paths between things in X that look like a space A:

Definition 13. For two maps $f, g : A \to X$, a homotopy from f to g is a continuous map $H : A \times I \to X$ where $H(a, 0) = f(a)$ and $H(a, 1) = g(A)$

Remark 14. One can generally describe the set of continuous functions $A \to X$ as a space, but actual paths in that space might not be the same as homotopies between those maps. But in HoTT we take a version of that equivalence as an axiom for function spaces (Function Extensionality).

We now have the tools to define paths between paths!

Definition 15. For two paths $\gamma, \sigma : I \to X$ between points x and y, a (2-dimensional) path from γ to σ is a homotopy from γ to σ *through paths from x to y*, meaning a map $\alpha : I \times I \to X$ restricting to a path from x to y on each $I \times \{t\}$, namely γ when $t = 0$ and σ when $t = 1$. (Point at picture)

Remark 16. This is when our groupoids become funkier. If morphisms are paths, what would be a path of paths? A morphism of morphisms? That's not a thing. So let's make it a thing! Call morphisms between morphisms 2-morphisms and write them as $H : f \implies g$ for (1-)morphisms f and g with the same source and target. For now we'll just let them be an abstract thing.

The first higher dimensional jump is always the hardest, so now we just extend this in each subsequent dimension.

Definition 17. If α and β are 2-paths between 1-paths γ and σ, a 3-path from α to β is a map $I^2 \times I \to X$ restricting to α, β, γ, and σ in the appropriate places. (Point at picture)

Remark 18. Here again, we will extend our notion of a groupoid to allow for 3-morphisms between parallel 2-morphisms, where the arrows now have 3 stems. (Point at picture)

I won't continue defining higher paths in each dimension all day; for each subsequent dimension increment each of the dimensions in the previous definition by one. It is worth mentioning that topologically, the union of two n-disks gluing along their (S^{n-1} shaped) boundaries is the n-sphere S^n, so as an n-path in X is a map $D^n \cong I^n \to X$ satisfying some boundary condition, an $n+1$-path takes two of these disks in X compatible on their boundaries and fills in the (map from the) sphere they form to get a ball (aka disk) of one dimension higher.

Remark 19. To summarize what we want to be able to do with our groupoids now, we have objects (0-morphisms?), (1-)morphisms between objects, 2-morphisms between parallel 1-morphisms, 3-morphisms between parallel 2-morphisms, and so on. These diagrams look a bit like gyroscopes and are sometimes called 'globes'. Note that unlike the 1-morphisms of a groupoid, I have yet to put any conditions or algebraic structure on these higher morphisms. That comes next.

3. True Paths Stick Together

Time to start doing some algebra! That means defining ways of combining paths with nice properties.

Definition 20. If α and β are n-paths from γ to σ and σ to τ respectively, form the 'composite' n-path $\alpha\beta$ from γ to τ by concatenation: $\alpha\beta$ acts on $I^{n-1} \times [0, \frac{1}{2}]$ as α scaled by $\frac{1}{2}$ in the last component and as β (scaled) on $I^{n-1} \times [\frac{1}{2}, 1]$. (Point at picture)

Example 21. For 1-paths this is just gluing the paths together at their common endpoints and reparameterizing the function to fit both into one.

Remark 22. For groupoids, this is just composition of morphisms! For 1-morphisms we already have this, as any two compatible morphisms in a category have a composite. For higher morphisms, this is something to add to our definition: n-morphisms $\alpha : \gamma \implies \sigma$ and $\beta : \sigma \implies \tau$ compose to form an n-morphism $\gamma \implies \tau$. (Point at picture)

So we now have a binary operation on n-paths, but we want it to be like a group(oid?), so we need identities and inverses.

Definition 23. For any n-path $\gamma : I^n \to X$ (where n can be 0 and γ then just a point), there is an $n+1$-path $const_\gamma$ given by $I^n \times I \to X : (a,t) \mapsto \gamma(a)$ ignoring the last I component.

Example 24. For a point x, $const_x$ is the constant function $I \to X$ sending all of I to x, for a 1-path γ, $const_\gamma$ is the 'degenerate' 2-path which looks like just γ.

Remark 25. For groupoids, the constant path at a point corresponds to the identity morphism of an object. Constant higher paths like $const_\gamma$ suggest that there ought to be identity $n+1$-morphisms for each n-morphism γ, which I'll write as id_γ. However, the topological motivation suggests something slightly weaker than saying these are identities for the composition operation.

Notice that for paths, concatenating γ with $const_x$ looks like γ from the outside, but has a different description as a map $I \to X$. However, the two are the same *up to homotopy* (draw picture), meaning they are connected by a higher path. In a similar vein, path composition is associative *only up to higher paths.* Now we just need inverses.

Definition 26. For an n-path $\gamma : I^{n-1} \times I$, define the n-path $\gamma^{-1} : I^{n-1} \times I : (a,t) \mapsto \gamma(a, 1-t)$ which follows γ but in the opposite direction.

It it easy to see that $\gamma\gamma^{-1}$ and vice versa are *homotopic to* constant n-paths via an $n+1$ path, so paths have inverses up to higher paths. (Point at picture)

Remark 27. How should we describe these structures for higher groupoids? Instead of requiring there to be identity morphisms and inverses to every morphism, we can impose these rules *up to higher morphisms*, so even for 1-morphisms we only require that $\gamma \circ id$ have a 2-morphism to γ and $\gamma\gamma^{-1}$ have a 2-morphism to id. We then impose the same *weak* groupoid structure for each level of morphisms: every n-morphism has an inverse up to some $(n+1)$-morphism, identity morphisms are only identities up to higher morphisms, and composition need only be associative up to higher morphisms. This is called a 'weak ∞-groupoid'.

4. Space Cats

Definition 28. This is when the worlds collide. Given a space X, we can form its 'fundamental ∞-groupoid' with the points of X as objects, paths between points as 1-morphisms, and n-paths as n-morphisms. I've essentially shown by definition that these paths have weak groupoid structure in every dimension, so this is a weak ∞-groupoid.

Example 29. Sometimes people talk about the 'fundamental groupoid' of a space, which is actually a real life groupoid. To get this from our ∞-groupoid, keep the objects the same (points in X) but take the morphisms between two points to be 'homotopy classes of paths' between them (it's easy to check that homotopy forms an equivalence relation on maps). Under this equivalence relation, what we get satisfies the associativity, identity, and inverse laws on the nose.

Example 30. Choose a 'basepoint' $x \in X$. The restriction of the fundamental groupoid of X to the object x and classes of paths from x to itself form a group called the 'fundamental group' of X based as x, written $\pi_1(X, x)$.

Example 31. Restricting the fundamental infinity groupoid to the object x, then further restricting to the 1-morphism $const_x$ and taking homotopy classes of 2-paths $const_x \implies const_x$ gives the 2nd homotopy group $\pi_2(X, x)$ of X at x, and so on for all π_n defines the 'higher homotopy groups'.

We can also start with a weak ∞-groupoid and get a CW complex:

Definition 32. Given a small weak ∞-groupoid G, its 'geometric realization' is the CW complex formed by adding a point for each object in G, a path ($\cong D^1$) between the appropriate points for each 1-morphism, a 2-path ($\cong D^2$) between the appropriate paths for each 2-morphism, and so on.

This construction follows the scheme I described above for picturing $(n+1)$-paths as filling in the spheres formed by pairs of compatible n-paths, where the attaching map glues half of the boundary of the $(n+1)$-disk to the first n-path and the other half to the second n-path.

Example 33. Draw stuff

Remark 34. In various different technical senses, weak ∞-groupoids capture all of the homotopy information of a space. HoTT is then useful for homotopy theory as it takes weak ∞-groupoids as the basic building blocks, and lots of nice spaces can be described by simple ∞-groupoids which are easy to work with.

5. All Good Things Shrink to an End

Now that we feel more comfortable with paths and homotopies, we can see some results that provide intuition for type theory.

Definition 35. Two spaces X and Y are homotopy equivalent if there are continuous maps $f : X \to Y$ and $g : Y \to X$ which are inverses up to homotopy.

Proposition 36. *The space $P(X, x)$ of paths in X starting at x is homotopy equivalent to the singleton $\{const_x\}$.*

Proof. (Sketch) We only really need to give a homotopy from the identity of $P(X, x)$ to the constant function on $P(X, x)$ at $const_x$, which means a function $H : P(X, x) \times I \to P(X, x)$. The idea is that for any path starting at x, we can shrink it back along itself all the way to $const_x$.

Explicitly, we would write $H(\gamma, t) : I \to X : s \mapsto (1-t)\gamma(s)$. When $t = 0$ this is just γ and when $t = 1$ this is $const_x$, and H is continuous as it does the same thing for each γ.

This gives the justification for the based path induction principle in type theory. For the regular path induction principle, let's demonstrate the following stronger version:

Proposition 37. *The space $P(X)$ of paths in X is homotopy equivalent to X.*

Proof. (Sketch) Consider the map $source : P(X) \to X$ sending a path γ to its first endpoint $\gamma(0)$ and $const : X \to P(X)$ sending a point x to the path $const_x$. The source of $const_x$ is clearly x, so the first composite is the identity on the nose. For the second composite, we need a homotopy from $id_P(X)$ to the map sending γ to $const_{\gamma(0)}$. But this is just the homotopy from the previous proof $H(\gamma, t) : I \to X : s \mapsto (1-t)\gamma(s)$, so we're done.

This proof is meant to give the intuition for the path induction principle: all paths can be continuously shrunk down to constant paths in the ambient space of all paths in the space.