Operations Arising from Equivariant K-theory of the Symmetric Group

Brandon Shapiro
K-Theory

Equivariant K-theory extends ideas from regular topological K-theory to spaces and vector bundles equipped with an action of a group, which in a natural way generalizes both topological K-theory and representation theory. Applying this theory to the symmetric group using examples defined from tensor product constructions allows us to derive the Adams operations from the representation theory of the symmetric group.

I will establish some basic properties of G bundles, make explicit the relationship betweek K-theory, representation theory, and equivariant K-theory in the case of spaces with trivial group action, then show how a simple example gives rise to a family of operations including the Adams operations.

1. G-Bundles

Definition 1. For a group G, a G-space is a space X with a homomorphism $G \to Homeo(X)$. I will write $g.x$ for the image of $g \in G$ under this homomorphism applied to $x \in X$. A morphism of G-spaces, or G-map, is a continuous map $f : X \to Y$ such that for each $g \in G$, $f(g.x) = g.f(x)$.

Definition 2. For X a G-space, a G-vector bundle on X is a G-space E equipped with a G-map $E \to X$ such that each fiber is a (complex) vector space and for all $g \in G$, the restriction of the associated homeomorphism to a map $E_x \to E_{g.x}$ on each fiber is a (necessarily iso-)morphism of vector spaces.

Example 3. For any G-module M, and X a G-space, $X \times M$ is a G-bundle, written $\mathbf{M}$, 'trivial' in the same sense as a trivial vector bundle over a topological space.

If X is a G-space with the trivial G-action (all $g \in G$ act on X by the identity function), then a G-bundle over X is simply a vector bundle on the space X with a G-action on each fiber. Each fiber is thus a 'G-module', a complex vector space with a G-action (or representation space for G).

Example 4. For any vector bundle E on a space X, the bundle $E^{\otimes k}$ has the structure of an S_k-vector bundle over the trivial S_k-space X, where the action on each fiber permutes the k components of the tensor product.

Definition 5. A homomorphism of G-vector bundles $E \to F$ is a G-map which restricts to a vector space homomorphism $E_x \to F_x$ on each fiber.

Example 6. For G bundles E and F, there is a G-bundle $Hom(E, F)$ where $Hom(E, F)_x = Hom(E_x, F_x)$. For the G-action, we define for $\phi : E_x \to F_x$ and $g \in G$, $g.\phi : E_{g.x} \to F_{g.x} : e \mapsto g.\phi(g^{-1}.e)$.

We can see that a homomorphism of bundles from E to F is precisely what is called an 'equivariant section' of $Hom(E, F)$, which is a G-map $X \to Hom(E, F)$ right inverse to the bundle map. To see this, consider an equivariant section consisting of continuously varying linear maps $s_x : E_x \to F_x$, where $g.s_x = s_{g.x}$. That is, for all $e \in E_{g.x}$, $(g.s_x)(e) = g.(s_x(g^{-1}.e)) = s_{g.x}(e)$. Swapping $g.e'$ in for e gives $g.s_x(e') = s_{g.x}(g.e)$, so the maps s_x assemble into a homomorphism $s : E \to F$ with $g.s(e) = s(g.e)$ for all $e \in E$. (The reverse argument works just as well, showing that the homomorphisms are exactly the equivariant sections of the Hom bundle.)

Proposition 7. *Let G be a compact group, X a trivial G-space, and E a G-vector bundle on X. Then there is an idempotent homomorphism $(-)^G : E \to E$ defined by $e^G = \frac{1}{|G|}\Sigma_{g\in G} g.e$ with image the sub-bundle of E fixed by G.*

I will henceforth assume that all groups G are compact (this will not be very restrictive).

Example 8. It immediately follows that if X is a trivial G-space then the fiber maps of homomorphisms $E \to F$ of bundles over X form a bundle $Hom_G(E, F) = (Hom(E, F))^G \subseteq Hom(E, F)$ over X. As the bundle has trivial G-action, it is sensible to consider it just as a regular vector bundle over the space X.

Proposition 9. *A homomorphism of G-bundles is an isomorphism if it restricts to an isomorphism on each fiber.*

2. K-Theory

Definition 10. The isomorphism classes of G-vector bundles on a G-space X form an abelian semigroup under direct sum, and we define $K_G(X)$ as its group completion.

I will write K for K^0 throughout this talk, for both equivariant and non-equivariant K-theory.

Example 11. $K_G(*) \cong R(G)$, the representation ring of G. A G-vector bundle over a point is just a G-module, which defines a representation of G, and the associated group completion is the underlying group of $R(G)$. For any G-space X, pullback along the unique (G-)map $X \to *$ defines a map $R(G) \cong K_G(*) \to K_G(X)$ which on bundles sends each G-module M to the trivial bundle $\mathbf{M}$.

Like the previous example, in many examples that follow I will describe operations on (G-)bundles then simply state that they extend to operations on the associated K groups. While I will not provide such proofs here, they are often nontrivial.

Example 12. If X is any space given the trivial G-action, there is a natural inclusion $K(X) \to K_G(X)$ induced by assigning the trivial G-action to any vector bundle on X. We also have the natural map $R(G) \to K_G(X)$ sending a representation space M to $X \times M$ with the G-action of M on each fiber. A G-bundle over the trivial G-space X consists of a vector-bundle over X (belonging to $K(X)$) and a G-action on each fiber (which can be described by an element of $R(G)$). This information uniquely determines the G-bundles over X in the following sense:

Proposition 13. *If X is a trivial G-space then the natural map $\mu : R(G) \otimes K(X) \to K_G(X)$ defined above is a ring isomorphism.*

Proof: (Sketch) It suffices to construct a map $\nu : K_G(X) \to R(G) \otimes K(X)$ inverse to μ. Let $\{M_i\}$ be a complete set of 'simple' G-modules, i.e. irreducible representations of G. For $[E] \in K_G(X)$, define $\nu([E]) = \Sigma_i [M_i] \otimes [Hom_G(\mathbf{M}_i, E)]$. Note that $\{M_i\}$ generates $R(G)$.

Applying $\mu \circ \nu$ to the class of a G-bundle E gives the class of the G-bundle $\bigoplus_i \mathbf{M}_i \otimes Hom_G(\mathbf{M}_i, E)$. As elements of the bundle $Hom_G(\mathbf{M}_i, E)$ are G-maps of the form $\mathbf{M}_x \to E_x$, there is a canonical map $\bigoplus_i \mathbf{M}_i \otimes Hom_G(\mathbf{M}_i, E) \to E$ (defined by the appropriate extensions of the evaluation map). On each fiber, the restriction map $\bigoplus_i M_i \otimes Hom(M_i, E_x) \to E_x$ is an isomorphism (this is a result from representation theory), so as discussed above the bundle map is an isomorphism. This shows that $\mu \circ \nu = id$.

In the other direction, for any simple G-module M_j and vector bundle F on X, $\nu \circ \mu$ sends $[M_j] \otimes [F]$ to $\Sigma_i [M_i] \otimes [Hom_G(\mathbf{M}_i, \mathbf{M}_j \otimes F)] = \Sigma_i [M_i] \otimes [Hom_G(\mathbf{M}_i, \mathbf{M}_j) \otimes F] = [M_j] \otimes [Hom_G(\mathbf{M}_j, \mathbf{M}_j \otimes F)] = [M_j] \otimes [F]$ as $Hom_G(\mathbf{M}_i, \mathbf{M}_j) \otimes F$ is F if $\mathbf{M}_i \cong \mathbf{M}_j$ and 0 otherwise. Extending this to all of $R(G) \otimes K(X)$ shows that $\nu \circ \mu) = id$.

Thus μ has an inverse and so is a bijection. It remains to show that the maps here extend to the K-groups and are ring isomorphisms, but I will not show that here. □

3. Operations

For any vector bundle E on a space X, the assignment $E \mapsto E^{\otimes k}$ as an S_k-bundle over X induces a natural transformation $K(X) \to K_{S_k}(X) \cong R(S_k) \otimes K(X)$. This is not not necessarily a group homomorphism, so we will only consider it a natural transformation of sets (also not obviously well-defined, see [2] chapter 2).

Given an element α of $R'_k := Hom(R(S_k), \mathbb{Z})$, we get a natural transformation $K(X) \xrightarrow{\otimes k} K_G(X) \xrightarrow{\nu} R(G) \otimes K(X) \xrightarrow{\alpha \otimes id} \mathbb{Z} \otimes K(X) \cong K(X)$. In the group $R(S_k) \otimes K(X)$, $[E^{\otimes k}]$ corresponds to $\Sigma_i [M_i] \otimes [Hom_{S_k}(\mathbf{M}_i, E^{\otimes k})]$, so it makes sense to think of α in terms of the values it takes on each $[M_i]$, the irreducible representations of S^k.

I will abuse notation slightly by writing $\alpha : K(X) \to K(X)$ for the operation determined by α in this manner.

Example 14. Let M be the trivial 1-dimensional representation of S_k. On each fiber, $Hom_{S_k}(M, E_x^{\otimes k})$ is the kth symmetric power of E_x, as each map is determined by the image of the single generator of M whose image must be S_k-invariant. If $\sigma^k \in R'_k$ sends M to 1 and all else to 0, $\sigma^k([E]) = [Hom_{S_k}(\mathbf{M}, E^{\otimes k})]$, which is the class of the kth symmetric power bundle of E (as the kth symmetric power on each fiber).

Example 15. Now let M be the 1-dimensional sign representation of S_k. As above, maps in $Hom_{S_k}(M, E_x^{\otimes k})$ are determined by the image of the generator, which must now be anti-symmetric. It follows that $\lambda^k \in R'_k$ sending M to 1 and all else to 0 acts on $[E]$ as the kth exterior power.

Let $\pi_M(E)$ denote $Hom_{S_k}(\mathbf{M}, E^{\otimes k})$, and let T_n be the diagonal matrix $(t_1, ..., t_n)$ considered as an endomorphism of $\mathbb{C}^n$. Then I claim that $Trace \pi_M(T_n)$ defines a symmetric polynomial in

$t_1, ..., t_n$ (as for any permutation matrix S we have $Trace\pi(T_n) = Trace\pi(S^{-1}T_nS)$). We then have a map

$$\Delta_{n,k} : R'_k \to Sym[t_1, ..., t_n] : \alpha \mapsto \Sigma_i \alpha(M_i) Trace\pi_{M_i}$$

Let $R'_* = \Sigma_k R'_k$. In the limit of n, we extend the maps $\Delta_{n,k}$ to a single map $\Delta : R'_* \to Sym$, the ring of symmetric functions.

Proposition 16. Δ *is an isomorphism of rings. (This suffices as a definition for a ring structure on* R'_**.)*

Example 17. $\Delta(\lambda^k) = e_k$, as $Trace\lambda^k(T_n) = e_k(t_1, ..., t_n)$ as the trace of the kth exterior power of T_n.

We can now (finally) define the Adams operations as $\psi_k = Q_k(\lambda^1, ..., \lambda^k) \in R'_*$, and it is clear (by definition in fact) that $\Delta(\psi^k)$ is the kth power sum symmetric function.

REFERENCES

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