Shape Independent Category Theory

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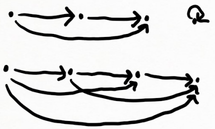
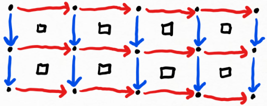
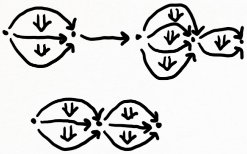
Category Theory OctoberFest 2019

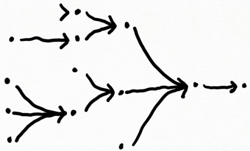
Categories with Different Cell Shapes

- Categories – dots, arrows
- 2-Categories – dots, arrows, 2-globs
- Double-Categories – dots, red/blue arrows, squares
- Multi-Categories – dots, n-to-1 arrows, $n \geq 0$



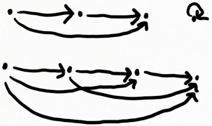


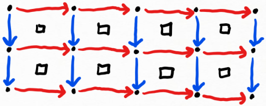


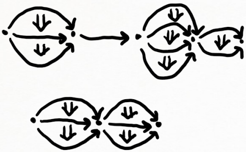


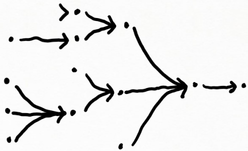
Nerves

- Categories $\to \widehat{\Delta}$ simplicial sets
- 2-Categories $\to \widehat{\Theta_2}$ Θ_2-sets
- Double-Categories $\to \widehat{\Delta \times \Delta}$ bisimplicial sets
- Multi-Categories $\to \widehat{\Omega}$ dendroidal sets





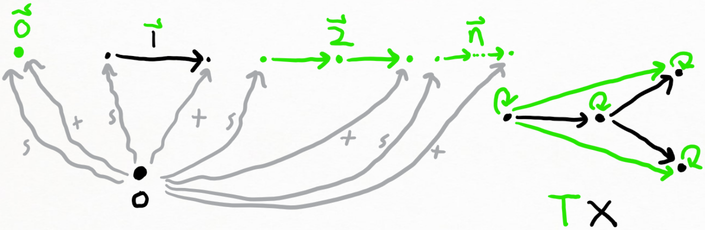




Familial Monads on Cell Diagrams

- G_1 is the category $0 \overset{s}{\underset{t}{\rightrightarrows}} 1$
- $\widehat{G_1}$ is the category of graphs
- Categories are algebras for a monad T on $\widehat{G_1}$
- $TX_0 = X_0 = Hom_{\widehat{G_1}}(\cdot, X)$

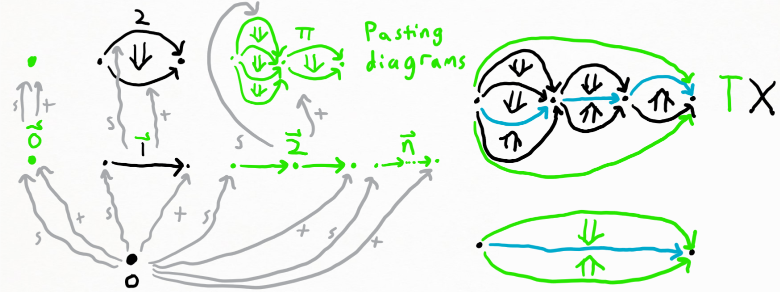
  $TX_1 = \{\text{paths in } X\} = \coprod_{n \geq 0} Hom_{\widehat{G_1}}(\cdot \to \overset{n}{\cdots} \to \cdot, X)$



Familial Monads on Cell Diagrams

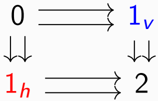
- G_2 is the category $0 \underset{t}{\overset{s}{\rightrightarrows}} 1 \underset{t}{\overset{s}{\rightrightarrows}} 2$
- $\widehat{G_2}$ is the category of 2-graphs
- 2-Categories are algebras for a monad T on $\widehat{G_2}$
- $TX_0 = Hom_{\widehat{G_2}}(\cdot, X) \quad TX_1 = \coprod_{n \geq 0} Hom_{\widehat{G_2}}(\cdot \to \overset{n}{\cdots} \to \cdot, X)$

  $TX_2 = \coprod_{\pi} Hom_{\widehat{G_2}}(\pi, X)$

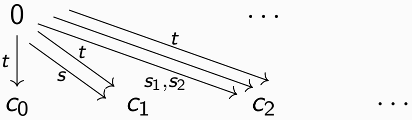


Familial Monads on Cell Diagrams

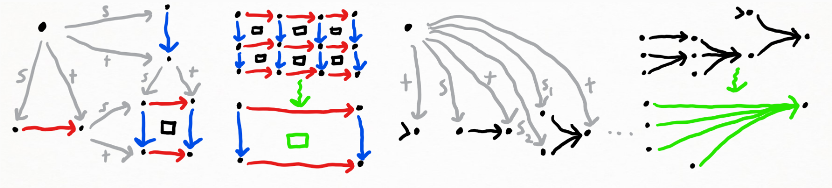
- $G_1 \times G_1$ is the category



- Double-Categories are algebras for a monad on $\widehat{G_1 \times G_1}$
- M is the category

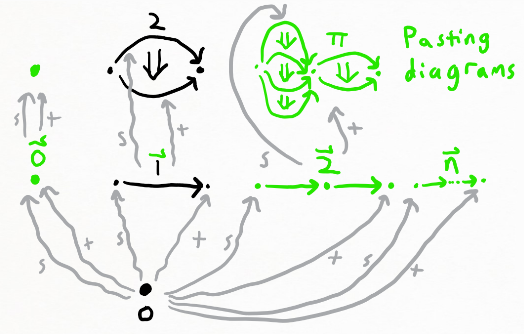


- Multi-Categories are algebras for a monad on $\widehat{M}$



Familial Monads on Cell Diagrams

- The data of a familial endofunctor F on $\hat{\mathcal{C}}$ consists of:
  - A functor $S : \mathcal{C}^{op} \to Set$
  - A functor $E : el(S) \to \hat{\mathcal{C}}$
- For c in $\mathcal{C}$, X in $\hat{\mathcal{C}}$, $FX_c = \coprod_{t \in Sc} Hom_{\hat{\mathcal{C}}}(Et, X)$



Theories and Nerves

- For each $t \in Sc$, an algebra A of T has a map

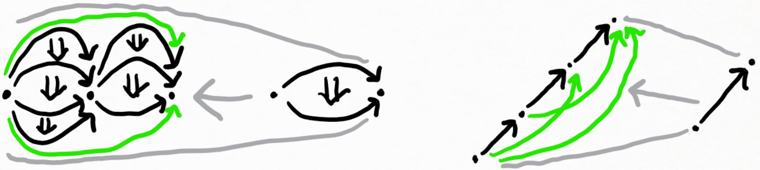
$$Hom_{\hat{\mathcal{C}}}(Et, A) \to A_c \cong Hom_{\hat{\mathcal{C}}}(y(c), A)$$

- This map is not representable, but its transpose is:

$$Hom_{TAlg}(TEt, A) \to Hom_{TAlg}(Ty(c), A)$$

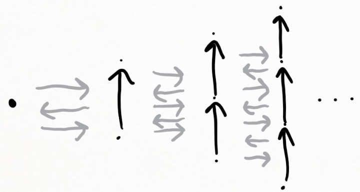
- The full subcategory $\mathcal{C}_T$ of $TAlg$ on $\{TEt\}$ has "cocomposition maps"
- (Weber 2007) The T nerve $N : TAlg \to \widehat{\mathcal{C}_T}$ is fully faithful:

$$NA_{TEt} = Hom_{TAlg}(TEt, A)$$



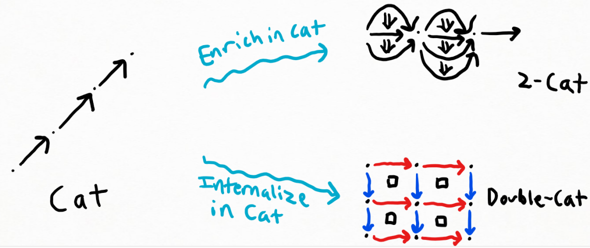
Theories and Nerves

- (Weber 2007) The T nerve $N : TAlg \to \widehat{\mathcal{C}_T}$ is fully faithful
- The full subcategory $\mathcal{C}_T$ of $TAlg$ on $\{TEt\}$ is the *theory* associated to T
- Nerves of T-algebras are functors $\mathcal{C}_T^{op} \to Set$ preserving certain limits
- Δ, Θ_2, Δ^2, and Ω all arise from this construction
- Those are all test categories...



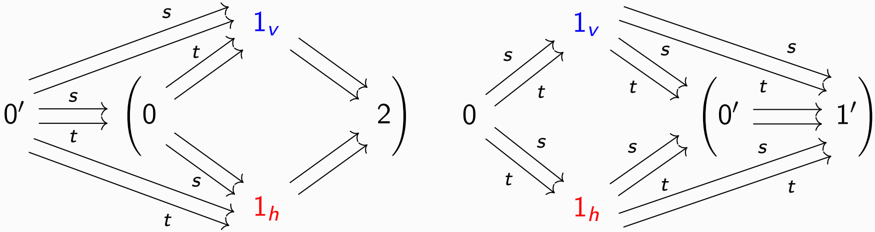
Shape Independent Category Theory

- Ideas from category theory should generalize to other familial algebras in cell diagrams (and often do!)
- Enriched categories are structures with new cell shapes
- So are internal categories
- These constructions extend to other familial representations



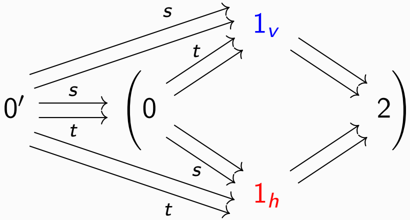
Enrichment via Cell Shapes

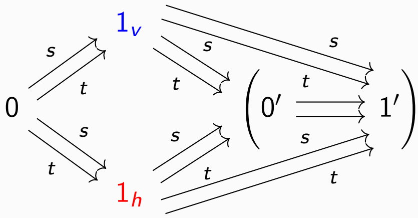
- Let $\mathcal{C}$ be a small direct category with local maximum object e
- For any small category $\mathcal{D}$, define $\mathcal{C} \wr_e \mathcal{D}$ to have:
  - Objects $ob(\mathcal{C})\backslash\{e\} \sqcup ob(\mathcal{D})$
  - Same morphisms $c \to c'$ as $\mathcal{C}$ and $d \to d'$ as $\mathcal{D}$
  - For all c, d, $Hom(c, d) = Hom(c, e)$, $Hom(d, c) = \emptyset$
  - For $f : c \to e$ in $\mathcal{C}$, $c \xrightarrow{f_d} d \xrightarrow{g} d' = c \xrightarrow{f_{d'}} d'$

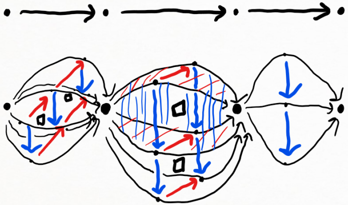


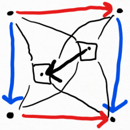
Enrichment via Cell Shapes

- Cell shapes of $\mathcal{C} \wr_e \mathcal{D}$ are e-cells stuffed with cell shapes of $\mathcal{D}$
- Cell diagrams in $\widehat{\mathcal{C} \wr_e \mathcal{D}}$ are cell diagrams over $\mathcal{C}$ stuffed with a diagram over $\mathcal{D}$ in each e-cell









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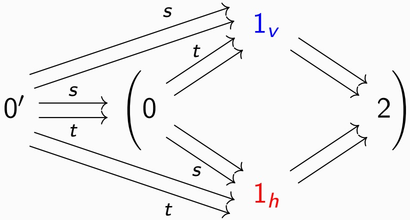
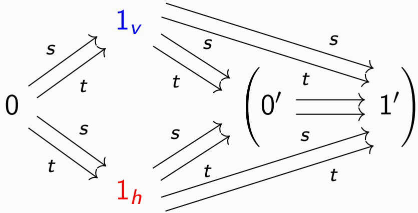
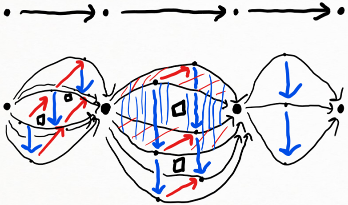
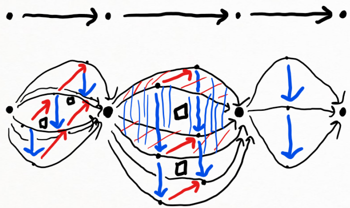






Enrichment via Cell Shapes

- Let $T_{\mathcal{C}}$, $T_{\mathcal{D}}$ be familial monads on $\widehat{\mathcal{C}}$, $\widehat{\mathcal{D}}$
- Build composable diagrams over $\mathcal{C} \wr_e \mathcal{D}$ by stuffing those over $\mathcal{C}$ with composable diagrams over $\mathcal{D}$
- (S.) These diagrams represent a familial monad T on $\widehat{\mathcal{C} \wr_e \mathcal{D}}$.
- (S.) When $\mathcal{C} = G_1$, T-algebras $\simeq$ $T_{\mathcal{D}}$-enriched categories.
- (S.) When $T_{\mathcal{C}}$ is "e-injective" and $T_{\mathcal{D}}$ "has enough degeneracies", the theory $(\mathcal{C} \wr_e \mathcal{D})_T \simeq \mathcal{C}_{T_{\mathcal{C}}} \wr \mathcal{D}_{T_{\mathcal{D}}}$ where $\mathcal{C}_{T_{\mathcal{C}}} \to \Gamma$ counts the e-cells in each $E_{\mathcal{C}} t$.



References

- Tom Leinster, *Higher Operads, Higher Categories*, London Mathematical Society Lecture Notes Series, Cambridge University Press, ISBN 0-521-53215-9.
- Mark Weber, *Familial 2-Functors and Parametric Right Adjoints*, Theory and Applications of Categories, Vol. 18, No. 22, 2007, pp. 665–732.

Thank You!