Familial Monads for Higher and Lower Category Theory

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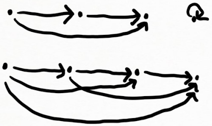
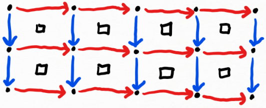
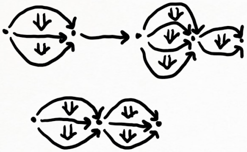
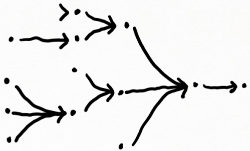
Categories with Different Cell Shapes

- Categories – dots, arrows
- 2-Categories – dots, arrows, globular 2-cells
- Double-Categories – dots, red/blue arrows, squares
- Multi-Categories – dots, n-to-1 arrows, $n \geq 0$



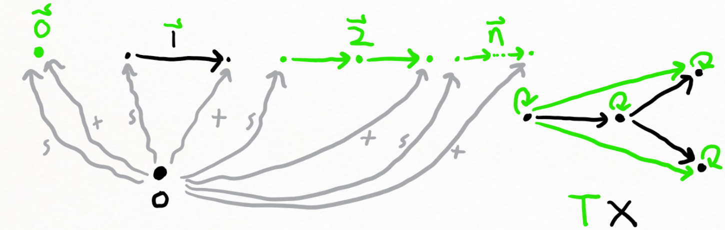






- G_1 is the category $0 \underset{t}{\overset{s}{\rightrightarrows}} 1$
- $\widehat{G_1} = Set^{G_1^{op}}$ is the category of graphs
- Categories are algebras for a monad T on $\widehat{G_1}$
- $TX_0 = X_0 = Hom_{\widehat{G_1}}(\cdot, X)$

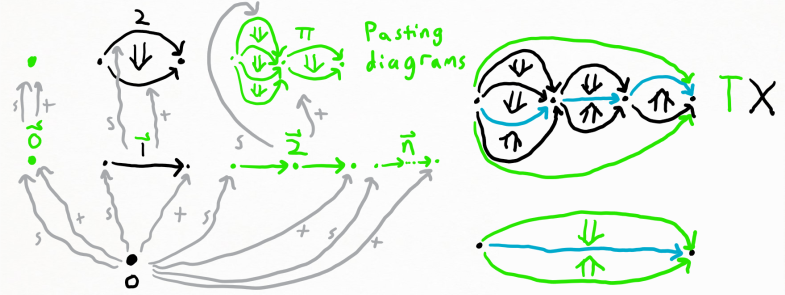
  $TX_1 = \{\text{paths in } X\} = \coprod_{n \geq 0} Hom_{\widehat{G_1}}(\cdot \to \overset{n}{\cdots} \to \cdot, X)$



Familial Monads on Cell Diagrams

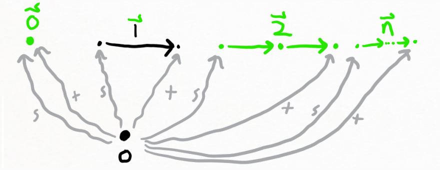
- G_2 is the category $0 \underset{t}{\overset{s}{\rightrightarrows}} 1 \underset{t}{\overset{s}{\rightrightarrows}} 2$
- $\widehat{G_2}$ is the category of 2-graphs
- 2-Categories are algebras for a monad T on $\widehat{G_2}$
- $TX_0 = Hom_{\widehat{G_2}}(\cdot, X) \quad TX_1 = \coprod_{n\geq 0} Hom_{\widehat{G_2}}(\cdot \to \overset{n}{\cdots} \to \cdot, X)$

  $TX_2 = \coprod_{\pi} Hom_{\widehat{G_2}}(\pi, X)$



Familial Monads on Cell Diagrams

- The data of a familial functor $F : \hat{\mathcal{D}} \to \hat{\mathcal{C}}$ consists of:
  - A functor $S : \mathcal{C}^{op} \to Set$ (operations outputting a c-cell)
  - A functor $E : \int S \to \hat{\mathcal{D}}$ (arities of the operations)
- For c in $\mathcal{C}$, X in $\hat{\mathcal{D}}$, $FX_c = \coprod_{t \in Sc} Hom_{\hat{\mathcal{D}}}(Et, X)$



Example: Free category monad on $\widehat{G_1}$

- $S0 = \{0\}$, $S1 = \mathbb{N}$, $\quad En = \quad \cdot \to \cdot \overset{n}{\cdots} \cdot \to \cdot$
- $TX_0 = Hom_{\widehat{G_1}}(\cdot, X)$, $TX_1 = \coprod_{n \geq 0} Hom_{\widehat{G_1}}(\cdot \to \cdot \overset{n}{\cdots} \cdot \to \cdot, X)$

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- For c in $\mathcal{C}$, X in $\hat{\mathcal{D}}$, $FX_c = \coprod_{t \in Sc} Hom_{\hat{\mathcal{D}}}(Et, X)$
- A monad (T, η, μ) on $\hat{\mathcal{C}}$ is familial if T is familial and η, μ are cartesian
- For 0 the empty category, a familial functor $\widehat{0} \to \hat{\mathcal{D}}$ is just a presheaf S over $\mathcal{D}$

Example: Free category monad on $\widehat{G_1}$

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- $TX_0 = Hom_{\widehat{G_1}}(\cdot, X)$, $TX_1 = \coprod_{n \geq 0} Hom_{\widehat{G_1}}(\cdot \to \overset{n}{\cdots} \to \cdot, X)$
- Unit and multiplication on edges given by length 1 paths and path concatenation

Familial Monads in *Poly*

- The category *Poly* of polynomial endofunctors on *Set* is a rich environment, including a monoidal structure $(\triangleleft, y)$ given by composition and identity
- Categories are $\triangleleft$ -comonoids in *Poly* (Ahman-Uustalu)
- Bicomodules in *Poly* from $\mathcal{D}^{op}$ to $\mathcal{C}^{op}$ are familial functors (aka prafunctors) $F : \hat{\mathcal{D}} \to \hat{\mathcal{C}}$ (Garner)
- Bicomodules from 0 to $\mathcal{D}^{op}$ are presheaves X over $\mathcal{D}$, and the composition $F \circ X$ of bicomodules is the presheaf FX over $\mathcal{C}$
- In the bicategory of categories and polynomial bicomodules, bimodules from the identity monad on $\widehat{0}$ to a familial monad T on $\hat{\mathcal{C}}$ are T-algebras
- In this sense, algebraic higher categories "live in" *Poly*

Commutativity Problems

- Familial endofunctors on *Set* are polynomial functors, of the form

$$FX = \coprod_{t \in S} Hom_{Set}(Et, X)$$

for some set S and functor $E : S \to Set$

- Monoids are algebras for a familial monad:

$$TX = \coprod_{n \in \mathbb{N}} Hom_{Set}(\underline{n}, X)$$

- The category of commutative monoids is not one of algebras for a familial monad:

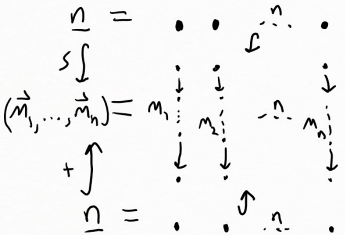
$$TX = \coprod_{n \in \mathbb{N}} Hom_{Set}(\underline{n}, X)/\Sigma_n$$

- Familial monads can't model strict commutativity conditions
- They can model commutativity *up to a higher cell*, like in symmetric monoidal categories

Free (Symmetric) Monoidal Categories on Graphs

- G_1 is the category $0 \underset{t}{\overset{s}{\rightrightarrows}} 1$, whose presheaves are graphs
- Define $S : G_1^{op} \to Set$ as $\mathbb{N} \underset{n \leftarrow\!\shortmid (m_1,\ldots m_n)}{\overset{n \leftarrow\!\shortmid (m_1,\ldots,m_n)}{\leftleftarrows}} \coprod_{n\in\mathbb{N}} \mathbb{N}^n$
- $E : \int S \to \widehat{G_1}$ sends n to $\underline{n}$ and $(m_1, \ldots, m_n)$ to $(\vec{m}_1, \ldots, \vec{m}_n)$ with the source and target inclusions as below
- The monad T on $\widehat{G_1}$ has strict monoidal cats as algebras:

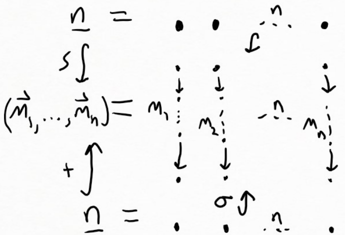
$$TX_0 = \coprod_{n\in\mathbb{N}} Hom_{\widehat{G_1}}(\underline{n}, X) \qquad TX_1 = \coprod_{n,m_1,\ldots,m_n\in\mathbb{N}} Hom_{\widehat{G_1}}((\vec{m}_1, \ldots, \vec{m}_n), X)$$



Free (Symmetric) Monoidal Categories on Graphs

- G_1 is the category $0 \underset{t}{\overset{s}{\rightrightarrows}} 1$, whose presheaves are graphs
- Define $S : G_1^{op} \to Set$ as $\mathbb{N} \underset{n \leftarrow\!\shortmid (m_1,\ldots m_n,\sigma)}{\overset{n \leftarrow\!\shortmid (m_1,\ldots,m_n,\sigma)}{\leftleftarrows}} \coprod_{n\in\mathbb{N}} \mathbb{N}^n \times \Sigma_n$
- $E : \int S \to \widehat{G_1}$ sends n to $\underline{n}$ and $(m_1, \ldots, m_n, \sigma)$ to $(\vec{m}_1, \ldots, \vec{m}_n)$ with the source and target inclusions as below
- T-algebras are now symmetric strict monoidal cats

$$TX_0 = \coprod_{n\in\mathbb{N}} Hom_{\widehat{G_1}}(\underline{n}, X) \qquad TX_1 = \coprod_{n,m_1,\ldots,m_n\in\mathbb{N},\sigma\in\Sigma_n} Hom_{\widehat{G_1}}((\vec{m}_1, \ldots, \vec{m}_n), X)$$



Free (Symmetric) Monoidal Categories on Graphs

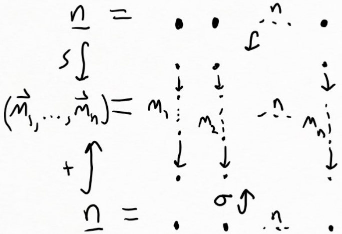
- T-algebras are now symmetric strict monoidal cats

$$TX_0 = \coprod_{n \in \mathbb{N}} Hom_{\widehat{G_1}}(\underline{n}, X) \qquad TX_1 = \coprod_{n, m_1, ..., m_n \in \mathbb{N}, \sigma \in \Sigma_n} Hom_{\widehat{G_1}}((\vec{m}_1, ..., \vec{m}_n), X)$$

- Write $v_1 \otimes \cdots \otimes v_n$ for the n-ary product of vertices $v_1, ..., v_n$ (aka $v : \underline{n} \to X$)
- When $m_1 = \cdots = m_n = 0$, T provides for $\sigma \in \Sigma_n$ an edge

$$v_1 \otimes \cdots \otimes v_n \to v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

- The case $m_1 = \cdots = m_n = 1$ encodes naturality of the symmetries, and the monad structure ensures invertibility etc.



Commutativity Solution?

- A discrete symmetric monoidal category is the same as a commutative monoid
- The category of commutative monoids is then the pullback of the diagram below

$$\begin{array}{ccc} & & \mathit{SymMonCat} \\ & & \downarrow \\ & & \widehat{G_1} & \xleftarrow[\mathit{discrete}]{} & \widehat{1} = \mathit{Set} \end{array}$$

- This diagram may be more easily described in *Poly* than the category of commutative monoids

References

- Brandon Shapiro, "Familial Monads as Higher Category Theories." arXiv:2111.14796
- David Spivak, "Functorial Aggregation." arXiv:2111.10968

Thanks!