Compositional Structure of Partial Evaluations

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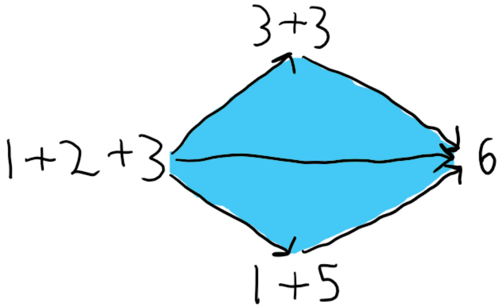
MIT Categories Seminar 9/10/20

Partial Evaluations

- Algebra is all about evaluating *formal expressions*
- Expressions can also be *partially evaluated*
- Partial evaluations form the paths in a directed space of formal expressions
- How does this space relate to algebra? Computation?



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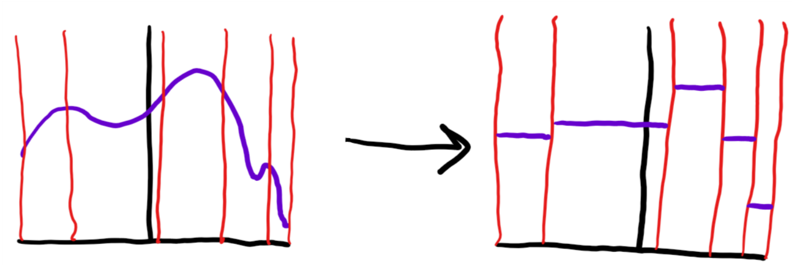
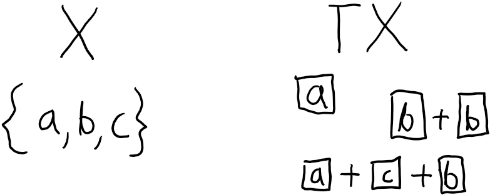
- Probability?



Monads

A *monad* is a functor $T : \mathcal{C} \to \mathcal{C}$ where TX describes formal expressions on X,

Example: “Free (commutative) monoid” monad



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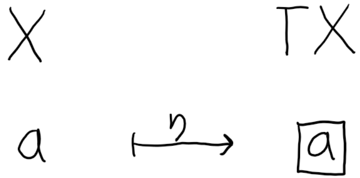

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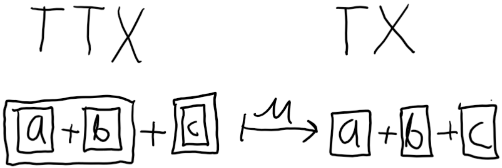

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- Unit and associativity equations:

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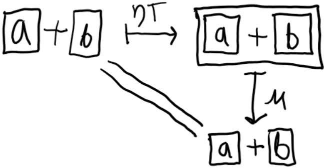

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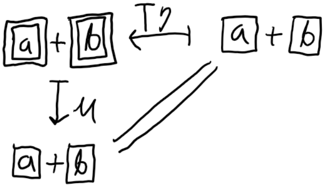

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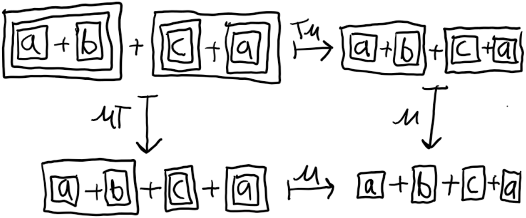

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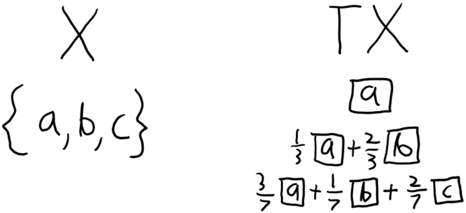
Example: Distribution monad



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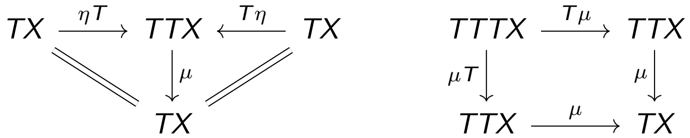

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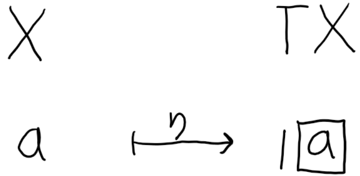

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$$\underbrace{\tfrac{1}{2}\boxed{\tfrac{1}{3}\boxed{a} + \tfrac{2}{3}\boxed{b}} + \tfrac{1}{2}\boxed{\tfrac{2}{3}\boxed{a} + \tfrac{1}{3}\boxed{c}}}_{TTX} \overset{\mu}{\longmapsto} \underbrace{\tfrac{1}{2}\boxed{a} + \tfrac{1}{3}\boxed{b} + \tfrac{1}{6}\boxed{c}}_{TX}$$

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& & TX & &
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$$

Example: Free S-module monad (S a semiring)

$$
\underbrace{\tfrac{1}{2}\boxed{\tfrac{1}{3}\boxed{a}+\tfrac{2}{3}\boxed{b}}+\tfrac{1}{2}\boxed{\tfrac{2}{3}\boxed{a}+\tfrac{1}{3}\boxed{c}}}_{TTX} \stackrel{\mu}{\longmapsto} \underbrace{\tfrac{1}{2}\boxed{a}+\tfrac{1}{3}\boxed{b}+\tfrac{1}{6}\boxed{c}}_{TX}
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Algebras

- An *algebra* for a monad T is an object A equipped with a map $e : TA \to A$ sending each formal expression to its *evaluation*

Example: (Commutative) monoid $\mathbb{N}$

$$T\mathbb{N} \qquad\qquad\qquad \mathbb{N}$$

$$\boxed{1} + \boxed{2} + \boxed{3} \quad \overset{e}{\longmapsto} \quad 1+2+3=6$$

Algebras

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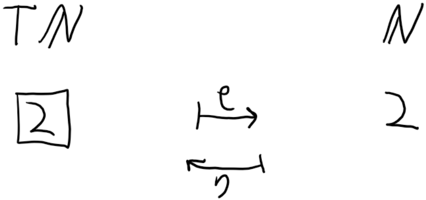

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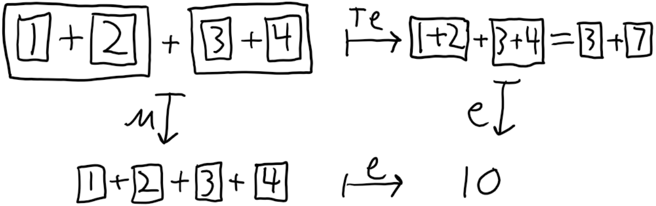

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$$

Example: Trivial S-module

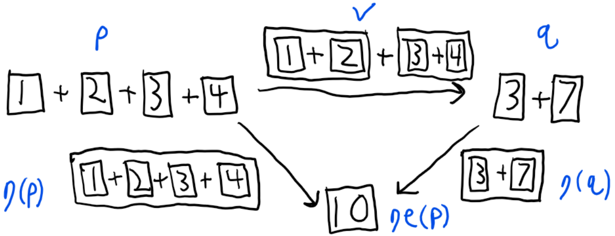
$$T\{*\} \cong S \qquad\qquad \{*\}$$

$$s\,\boxed{*} \quad \xmapsto{e} \quad *$$

Partial evaluations

- Consider a T-algebra (A, e) and formal expressions $p, q \in TA$
- A *partial evaluation* from p to q is a doubly nested expression $v \in TTA$ with $\mu(v) = p$ and $Te(v) = q$
- If p partially evaluated to q, then $e(p) = e(q)$
- There is always a partial evaluation $\eta(p)$ from p to $\eta e(p)$

Example: (Commutative) monoid $\mathbb{N}$



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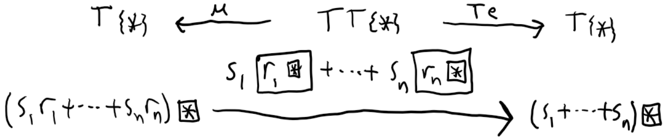
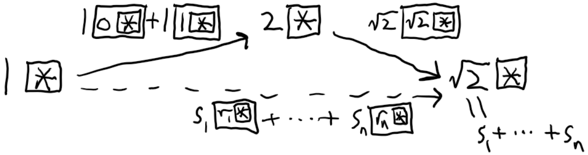


- Let $S = \mathbb{N}[\sqrt{2}] = \{n + m\sqrt{2}\}$



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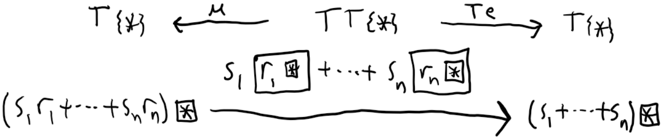

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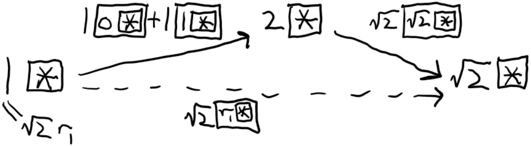

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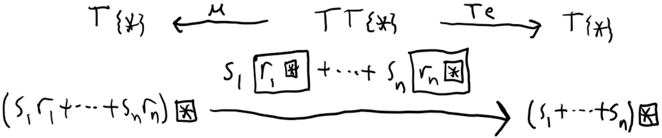

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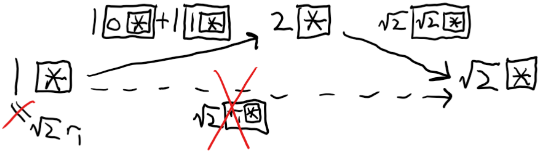

- (CFPS) Partial evaluations don't always compose

Bar Construction

- Partial evaluations fit into a richer structure, called the *Bar Construction* of a T-algebra A
- Relations between monad and algebra maps...

$$TA \xleftarrow{\mu} T^2A \xrightarrow{Te} TA$$

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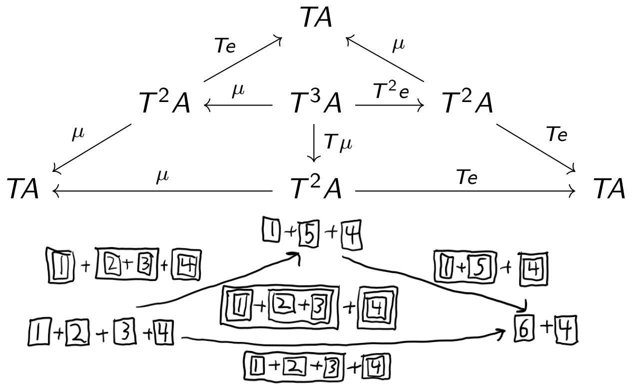

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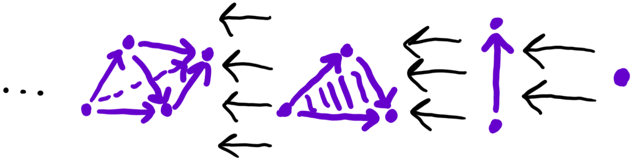
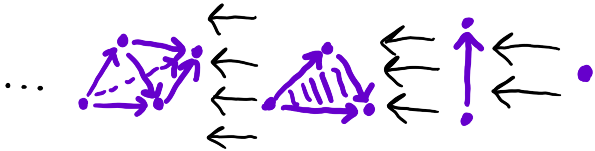


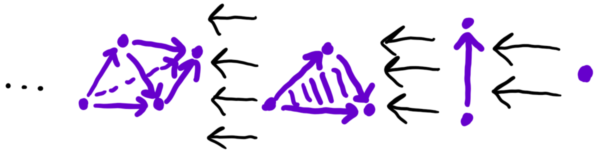
$$\cdots T^4A \underset{\mu}{\overset{T^3e}{\underset{T\mu}{\overset{T^2\mu}{\rightrightarrows\rightrightarrows}}}} T^3A \;\overset{T^2e,\; T\mu,\; \mu}{\rightrightarrows\!\!\to}\; T^2A \underset{\mu}{\overset{Te}{\rightrightarrows}} TA$$

...are given by the *simplicial identities*.



$$\cdots \{\text{3-simplices}\} \overset{d_0, d_1, d_2, d_3}{\rightrightarrows\rightrightarrows} \{\text{2-simplices}\} \overset{d_0, d_1, d_2}{\rightrightarrows\rightarrow} \{\text{1-simplices}\} \overset{d_0, d_1}{\rightrightarrows} \{\text{0-simplices}\}$$

Simplicial Sets



$$\cdots\; T^4A \;\underset{T\eta}{\overset{\rightarrow\rightarrow\rightarrow\rightarrow}{\underset{T^2\eta}{\underset{T^3\eta}{\leftarrow\leftarrow\leftarrow}}}}\; T^3A \;\overset{\rightarrow\rightarrow\rightarrow}{\underset{T^2\eta,\ T\eta}{\leftarrow\leftarrow}}\; T^2A \;\overset{\rightarrow\rightarrow}{\underset{T\eta}{\leftarrow}}\; TA$$

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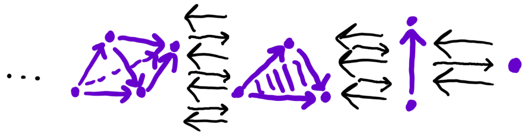
- The *simplex category* Δ is the category of finite nonempty ordered sets and order preserving functions.



- A *simplicial object* X in a category $\mathcal{C}$ is a functor $\Delta^{op} \to \mathcal{C}$.

$$\cdots X_3 \;\begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \\ \xrightarrow{d_2} \\ \xrightarrow{d_3} \\ \xleftarrow{s_0} \\ \xleftarrow{s_1} \\ \xleftarrow{s_2} \end{array}\; X_2 \;\begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \\ \xrightarrow{d_2} \\ \xleftarrow{s_0} \\ \xleftarrow{s_1} \end{array}\; X_1 \;\begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \\ \xleftarrow{s_0} \end{array}\; X_0$$

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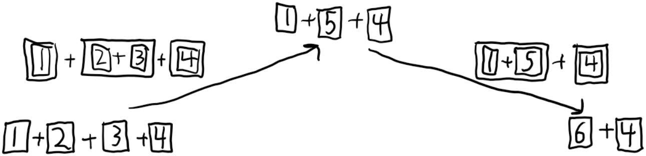
$$\cdots\; T^4A \;\underset{\substack{\xleftarrow{T^3\eta} \\ \xleftarrow{T^2\eta} \\ \xleftarrow{T\eta}}}{\overset{\substack{\xrightarrow{T^3e} \\ \xrightarrow{T^2\mu} \\ \xrightarrow{T\mu}}}{\xrightarrow{\mu}}}\; T^3A \;\underset{\substack{\xleftarrow{T^2\eta} \\ \xleftarrow{T\eta}}}{\overset{\substack{\xrightarrow{T^2e} \\ \xrightarrow{T\mu}}}{\xrightarrow{\mu}}}\; T^2A \;\underset{\xleftarrow{T\eta}}{\overset{\xrightarrow{Te}}{\xrightarrow{\mu}}}\; TA$$

- 1-simplices in this simplicial set are partial evaluations:



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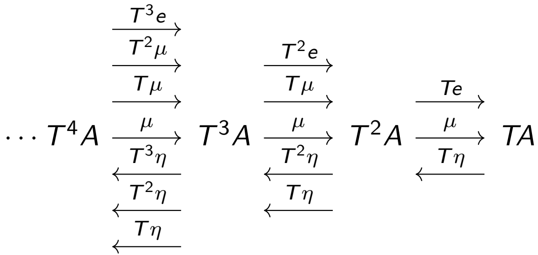
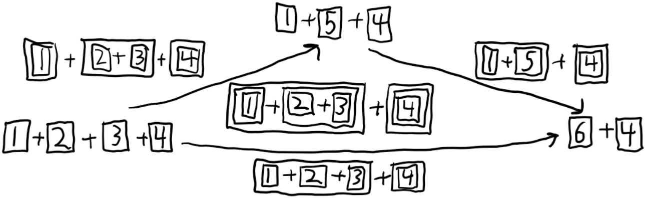


- 2-simplices in this simplicial set are "composition strategies":



Compositions

- When do successive partial evaluations have a composition strategy?

- Partial evaluations are equivalently maps $\{*\} \to T^2A$

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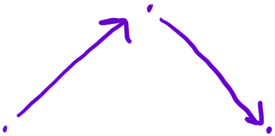

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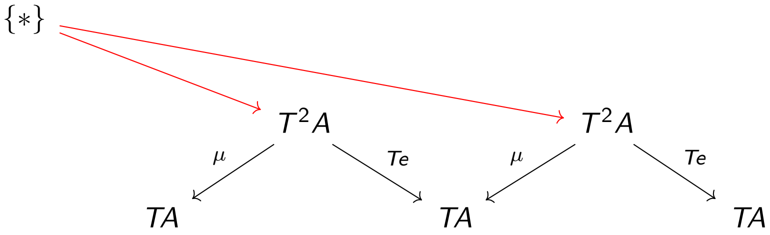

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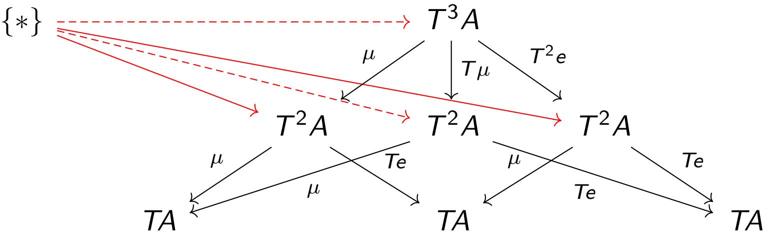
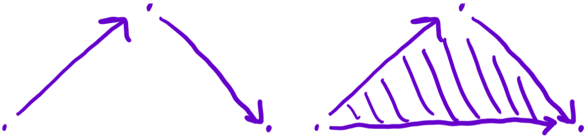
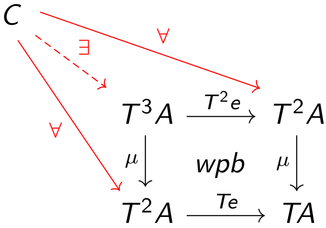
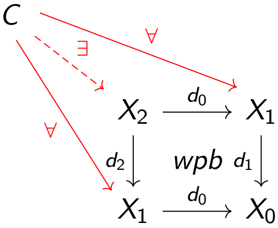


Compositions

- If the square is a *weak pullback* (aka *weakly cartesian*), the dashed map always exists but not necessarily uniquely
- In a simplicial set X, this property corresponds to having all inner 2-horn fillers

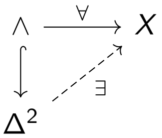


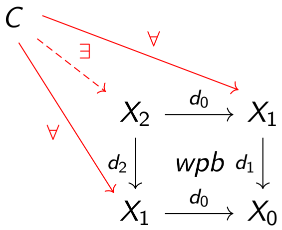




Compositions

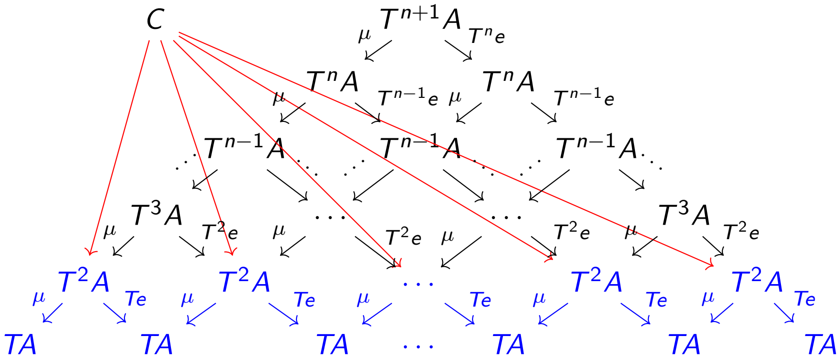
- If the square is a *weak pullback* (aka *weakly cartesian*), the dashed map always exists but not necessarily uniquely
- In a simplicial set X, this property corresponds to having all inner 2-horn fillers
- If the square is a (strong) pullback, the fillers are unique
- When is $Bar_T(A)$ the nerve of a category? A quasicategory?





Compositions

- When do partial evaluations form a category?
- If the naturality squares of μ are cartesian
- For $X = Bar_T(A)$, this means $X_n \cong X_1 \times_{X_0} \overset{n}{\cdots} \times_{X_0} X_1$



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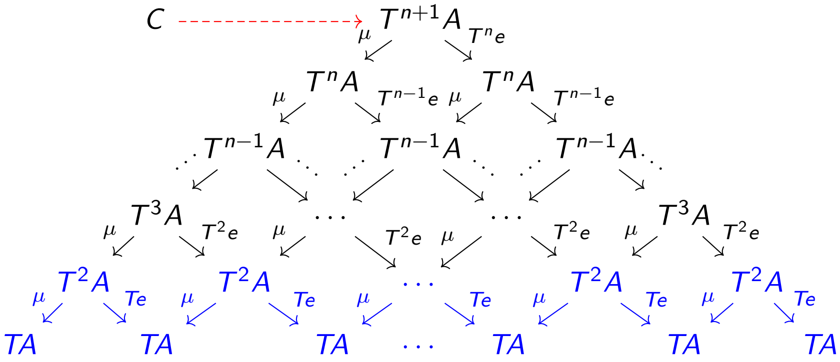
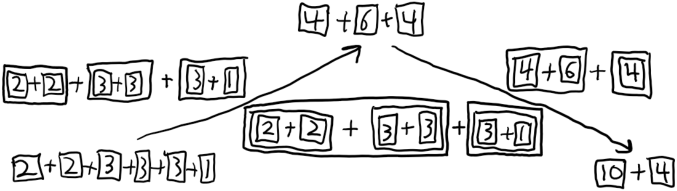
- For $X = Bar_T(A)$, this means $X_n \cong X_1 \times_{X_0} \overset{n}{\cdots} \times_{X_0} X_1$
- This makes $Bar_T(A)$ the nerve of a category with formal expressions as objects and partial evaluations as morphisms



- Free monoid monad (or any plain operad) has cartesian μ
- Free comm. monoid monad T has only weakly cartesian μ



$$\begin{array}{ccc} T^3A & \xrightarrow{T^2e} & T^2A \\ {\scriptstyle\mu}\downarrow & wpb & \downarrow{\scriptstyle\mu} \\ T^2A & \xrightarrow{Te} & TA \end{array}$$

BC Monads

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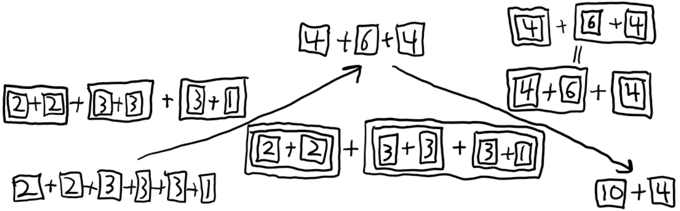

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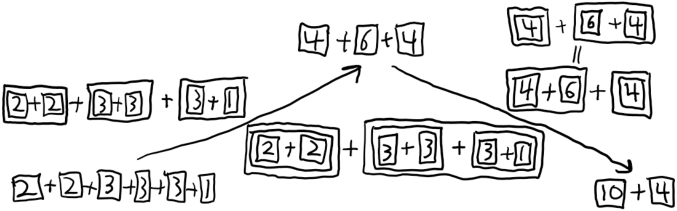
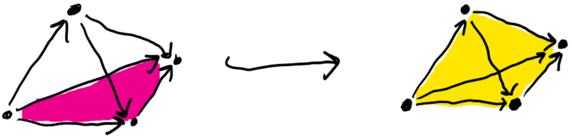


$$\begin{array}{ccc} T^{m+3}A & \xrightarrow{T^{m+2}e} & T^{m+2}A \\ \mu\downarrow & wpb & \downarrow\mu \\ T^{m+2}A & \xrightarrow{T^{m+1}e} & T^{m+1}A \end{array} \qquad \begin{array}{ccc} T^{m+4}A & \xrightarrow{T^{m+2}\mu} & T^{m+3}A \\ \mu\downarrow & wpb & \downarrow\mu \\ T^{m+3}A & \xrightarrow{T^{m+1}\mu} & T^{m+2}A \end{array}$$

BC Monads

- Free monoid monad (or any plain operad) has cartesian μ
- Free comm. monoid monad T has only weakly cartesian μ
- T also preserves weak pullbacks
- Such *BC monads* include distribution, any symmetric operad
- (CFPS) $Bar_T(\mathbb{N})$ is not a quasicategory



$$\begin{array}{ccc} T^{l+m+3}A & \xrightarrow{T^{l+m+2}e} & T^{l+m+2}A \\ {\scriptstyle T^l\mu}\downarrow & wpb & \downarrow{\scriptstyle T^l\mu} \\ T^{l+m+2}A & \xrightarrow{T^{l+m+1}e} & T^{l+m+1}A \end{array} \qquad \begin{array}{ccc} T^{l+m+4}A & \xrightarrow{T^{l+m+2}\mu} & T^{l+m+3}A \\ {\scriptstyle T^l\mu}\downarrow & wpb & \downarrow{\scriptstyle T^l\mu} \\ T^{l+m+3}A & \xrightarrow{T^{l+m+1}\mu} & T^{l+m+2}A \end{array}$$

Filler Conditions

- What properties does $Bar_T(A)$ have when T is BC?
- Let $n \geq 2$, $j - i > 1$
- A simplicial set X with this property

$$
\begin{array}{ccc}
T^{n+1}A & \xrightarrow{T^n e} & T^n A \\
{\scriptstyle T^{n-j}\mu}\downarrow & wpb & \downarrow{\scriptstyle T^{n-j}\mu} \\
T^n A & \xrightarrow[T^{n-1}e]{} & T^{n-1}A
\end{array}
\qquad
\begin{array}{ccc}
T^{n+1}A & \xrightarrow{T^{n-i}\mu} & T^n A \\
{\scriptstyle T^{n-j}\mu}\downarrow & wpb & \downarrow{\scriptstyle T^{n-j}\mu} \\
T^n A & \xrightarrow[T^{n-i-1}\mu]{} & T^{n-1}A
\end{array}
$$

Filler Conditions

- What properties does $Bar_T(A)$ have when T is BC?
- Let $n \geq 2$, $j - i > 1$
- A simplicial set X with this property is *inner span complete*

$$\begin{array}{ccc} X_n & \xrightarrow{d_i} & X_{n-1} \\ {\scriptstyle d_j}\downarrow & wpb & \downarrow{\scriptstyle d_{j-1}} \\ X_{n-1} & \xrightarrow[d_i]{} & X_{n-2} \end{array}$$

$$\begin{array}{ccc} \Delta^{n-1} \sqcup_{\Delta^{n-2}} \Delta^{n-1} & \xrightarrow{\forall} & X \\ \downarrow & \overset{\exists}{\nearrow} & \\ \Delta^n & & \end{array}$$

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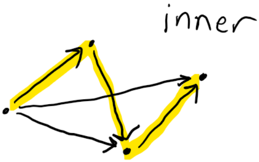

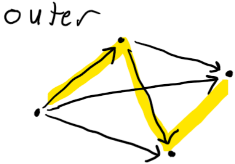

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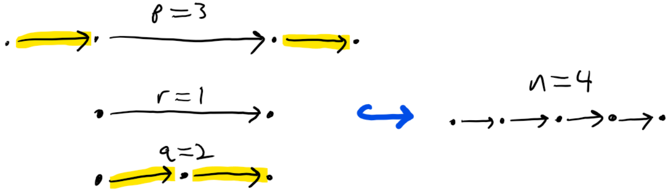
Filler Conditions

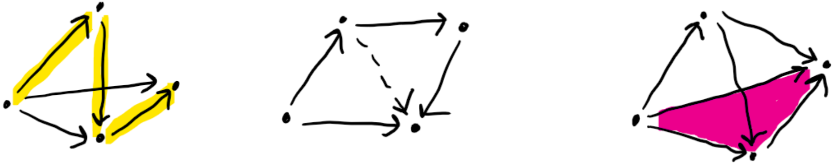
- What properties does $Bar_T(A)$ have when T is BC?
- Let $n \geq 2$, $j - i > 1$
- A simplicial set X with this property is *inner span complete*
- (CFPS) X then has fillers for all spans containing the spine



$$\begin{array}{ccc} X_n & \longrightarrow & X_p \\ \downarrow & wpb? & \downarrow \\ X_q & \longrightarrow & X_r \end{array} \qquad \begin{array}{ccc} \Delta^p \sqcup_{\Delta^r} \Delta^q & \xrightarrow{\forall} & X \\ \big\downarrow & \overset{\exists?}{\nearrow} & \\ \Delta^n & & \end{array}$$

Filler Conditions

- What other fillers do inner span complete simplicial sets have?
- (CFPS) All *directed acyclic configurations* $S \subset \Delta^n$:
  - S contains the spine of Δ^n
  - The 1-skeleton of S is *chordal*
  - S has $\partial\Delta^k \hookrightarrow \Delta^k$ fillers for $2 \leq k \leq n$
- Does not include any horns



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  - The 1-skeleton of S is *chordal*
  - S has $\partial\Delta^k \hookrightarrow \Delta^k$ fillers for $2 \leq k \leq n$
- Does not include any horns
- Includes spine inclusions and 2-Segal inclusions

Parting Thoughts...

- We can also describe when partial evaluations do or don't have inverses
- Inner span completeness is not a homotopical property
- How do properties of $Bar_T(A)$ relate to computation?
- Higher order rewriting?

Thank you!

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- Tobias Fritz and Paolo Perrone. Monads, partial evaluations, and rewriting. *Proceedings of MFPS 36, ENTCS*, 2020.
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