Categorical Tiling Theory

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2024 UVA Topology REU

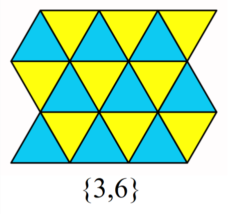
2025 Category Theory Octoberfest

Regular tilings of the plane

Euclidean tilings:

Regular tilings of the plane

Euclidean tilings:



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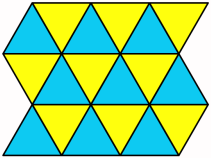

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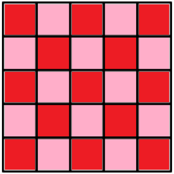

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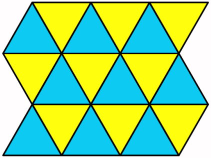

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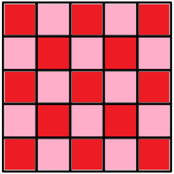

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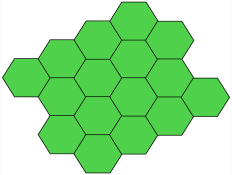

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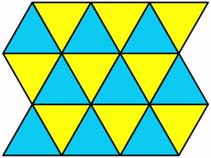

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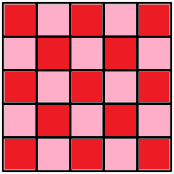

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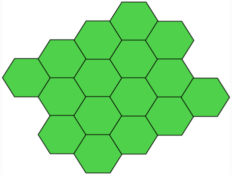

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Hyperbolic Tilings:

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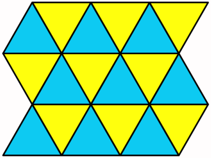

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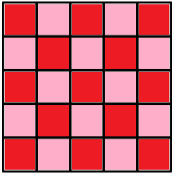

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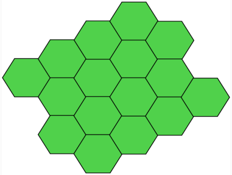

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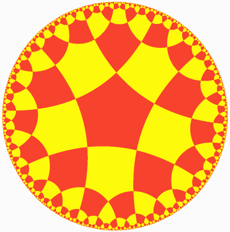

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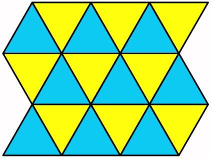

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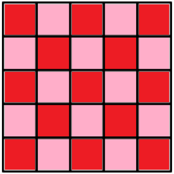

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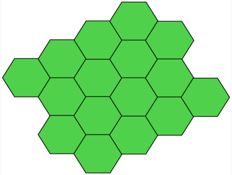

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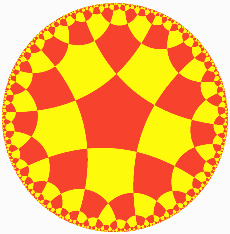

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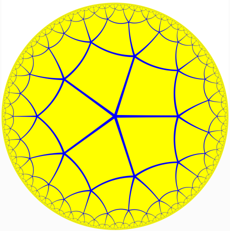

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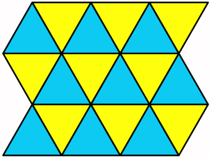

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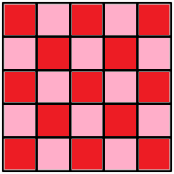

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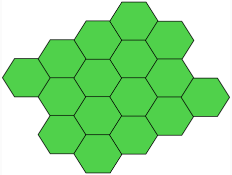

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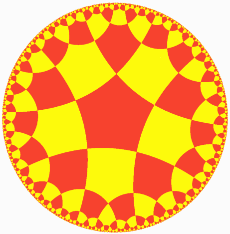

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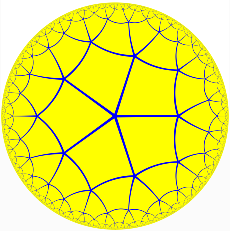

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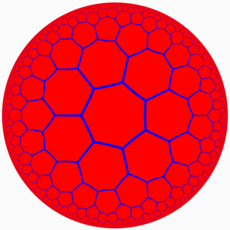
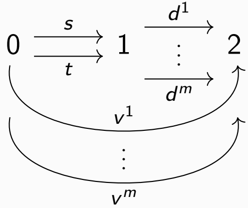


{7,3}

m-gon Categories as Directed tiles

An *m-gon category* has objects and non-identity mophisms

$\mathcal{C}_m :=$



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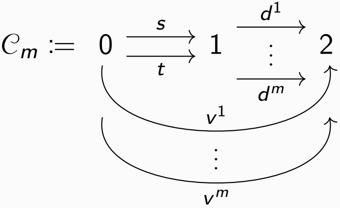
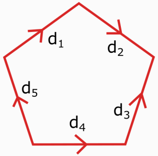


This corresponds to an m-gon with directed and labeled edges



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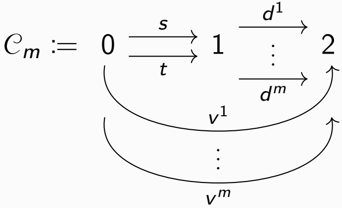

such that $\{d^i \circ s, d^i \circ t\} = \{v^i, v^{i+1 \pmod m}\}$ for $i = 1, \ldots, m$.

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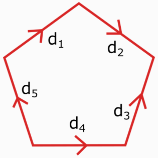

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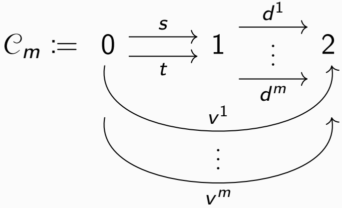

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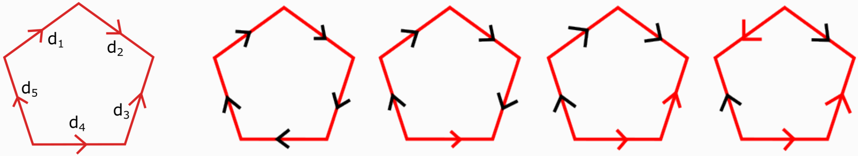
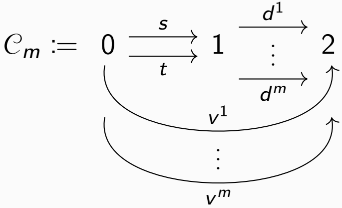


m-gon Categories as Directed tiles

A presheaf on an m-gon category contains vertices, edges, and tiles



$$\{\text{vertices}\} \overset{s}{\underset{t}{\leftleftarrows}} \{\text{edges}\} \overset{d_1}{\underset{d_5}{\leftleftarrows\leftleftarrows\leftarrow}} \{\text{tiles}\}$$

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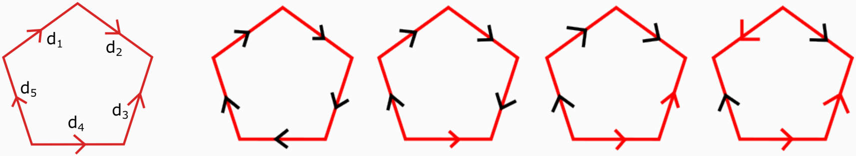

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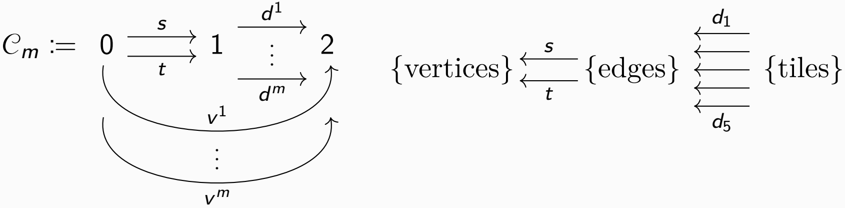

such that $\{s(d_i(x)), t(d_i(x))\} = \{v_i, v_{i+1 \pmod m}\}$ for each tile x.

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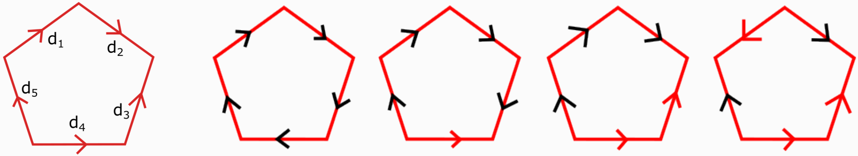

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$$\mathcal{C}_m := \quad 0 \underset{t}{\overset{s}{\rightrightarrows}} 1 \overset{d^1}{\underset{d^m}{\underset{\vdots}{\longrightarrow}}} 2, \qquad v^1, \dots, v^m : 0 \to 2$$

$$\{\text{vertices}\} \underset{t}{\overset{s}{\leftleftarrows}} \{\text{edges}\} \overset{d_1}{\underset{d_5}{\Lleftarrow}} \{\text{tiles}\}$$

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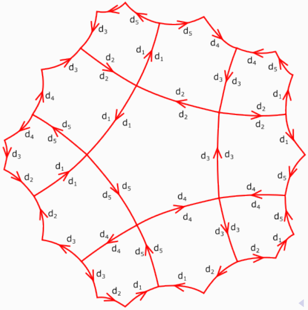

Directed tilings

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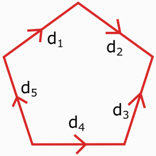

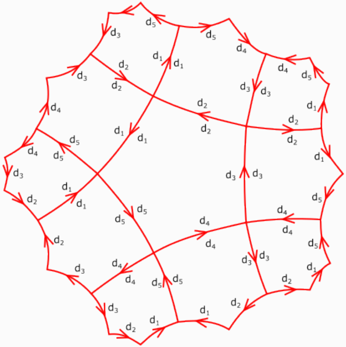

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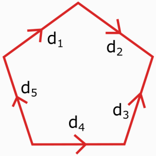

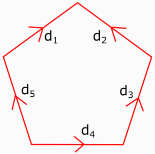

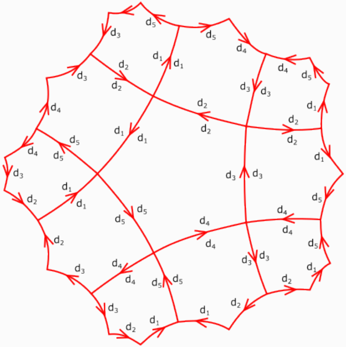

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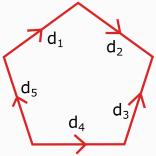
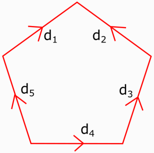
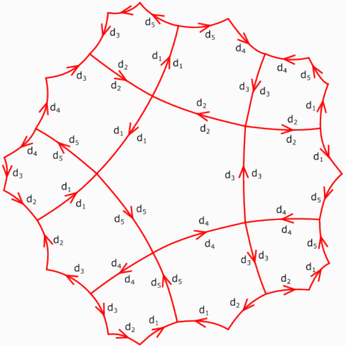
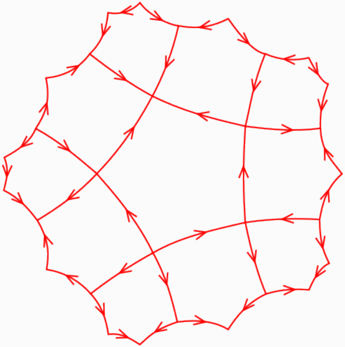


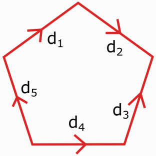


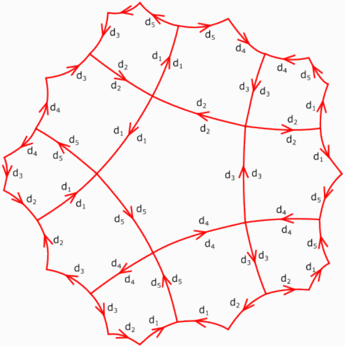




Reflective tilings

Whenever n is even, there is a reflective directed $\{m, n\}$ tiling:





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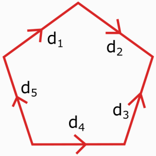

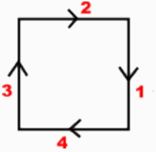

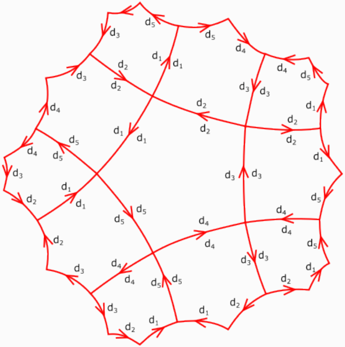

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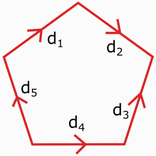
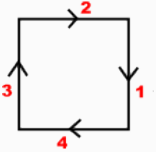
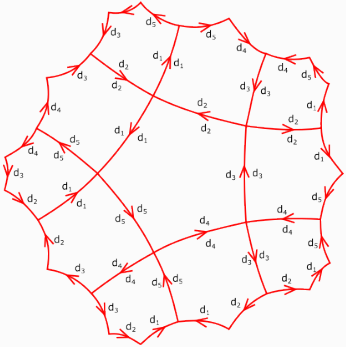
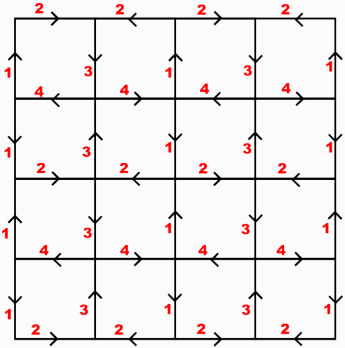


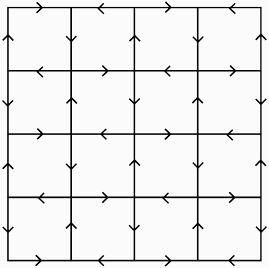






Reflection generated tilings

How to change the directions of a reflective tiling:



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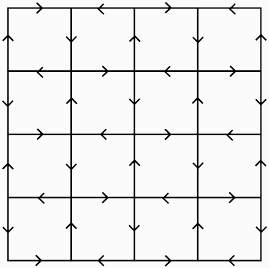

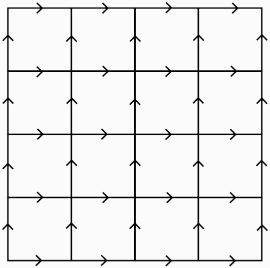

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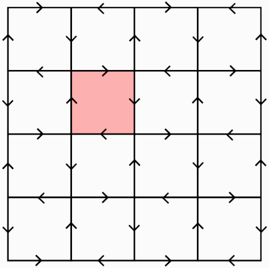

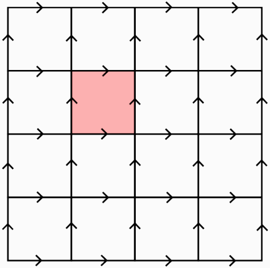

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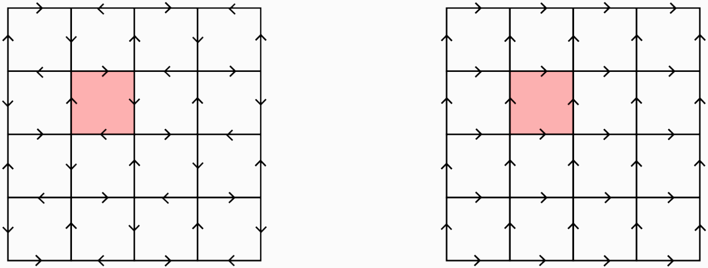
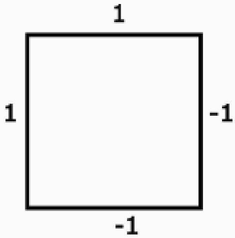


Reverse some edges of one tile:



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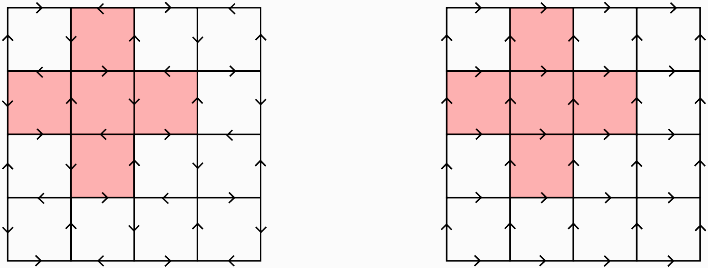

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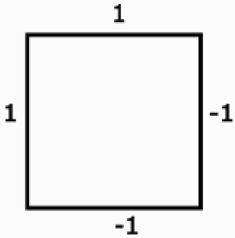

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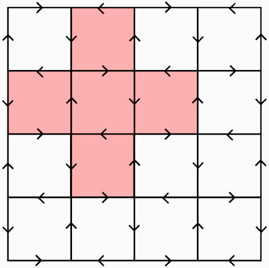
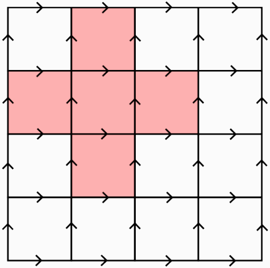
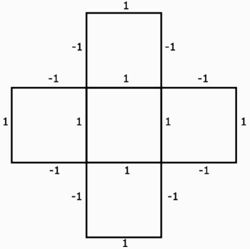
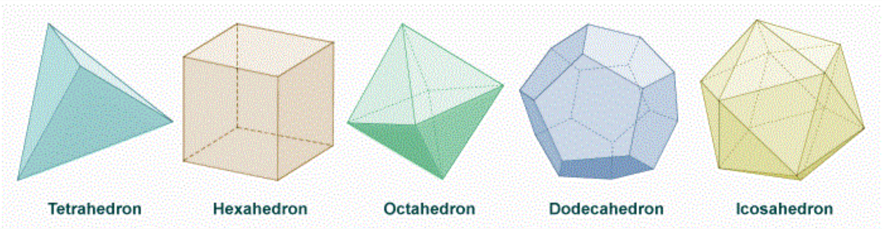




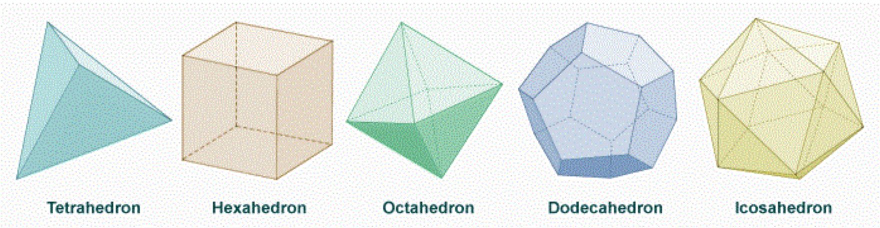
Reflect that tile outward and reverse more edges:



Higher Dimensions

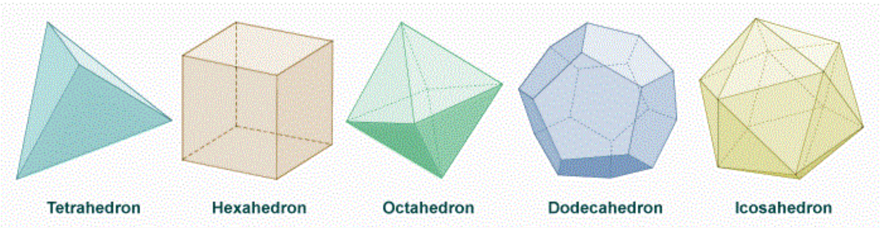


Higher Dimensions



Cube categories: $$0 \overset{s}{\underset{t}{\rightrightarrows}} 1 \;\overset{d^1}{\underset{d^4}{\vdots}}\; 2 \;\overset{d^1}{\underset{d^6}{\vdots}}\; 3$$

Higher Dimensions



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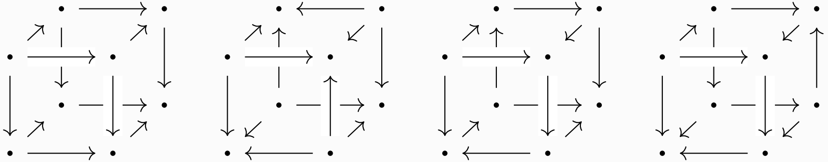
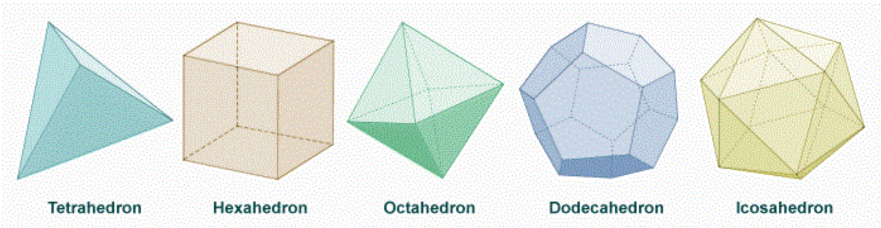
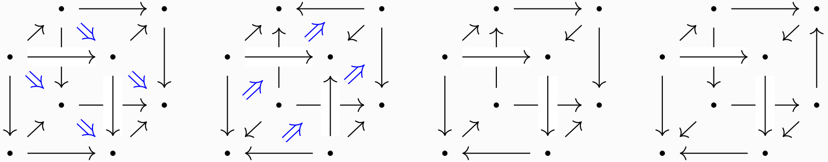


Higher Dimensions



$$\text{Cube categories:} \quad 0 \underset{t}{\overset{s}{\rightrightarrows}} 1 \underset{d^4}{\overset{d^1}{\vdots}} 2 \underset{d^6}{\overset{d^1}{\vdots}} 3$$



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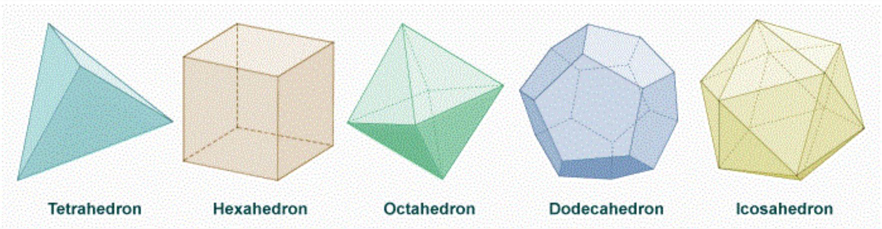

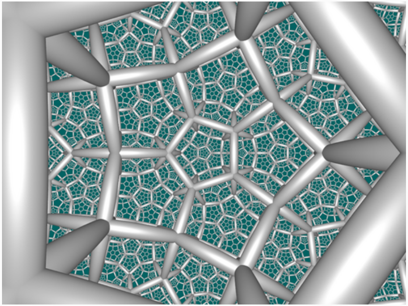

Subdivisions and Cat$^{\#}$

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Subdivisions and $\mathbf{Cat}^{\#}$

$$\mathcal{C}_{\mathsf{6}} \xleftarrow{\text{discrete fibration}} \overline{\mathcal{C}_{\mathsf{6}}} \longrightarrow \widehat{\mathcal{C}_{\mathsf{5}}}$$

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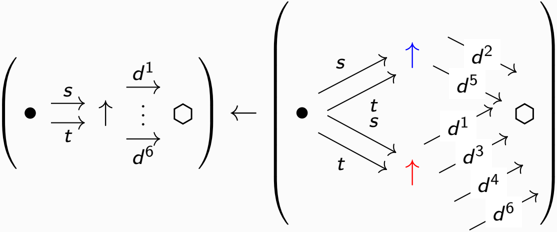

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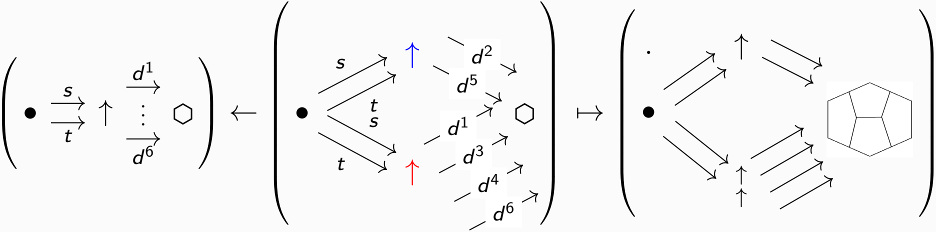

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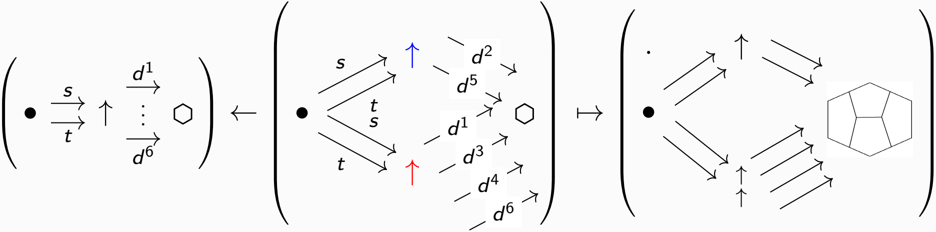

This is the data of a parametric right adjoint functor $\widehat{\mathcal{C}_6} \leftarrow \widehat{\mathcal{C}_5}$.

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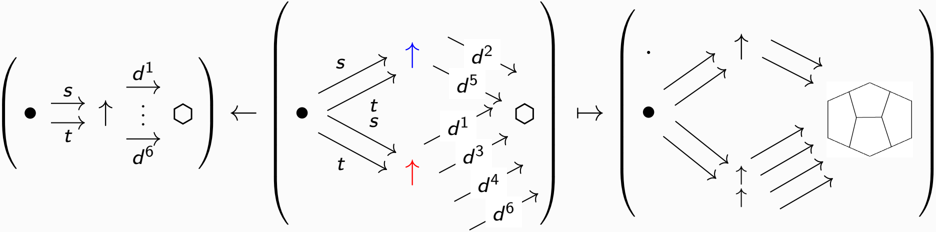

This is the data of a parametric right adjoint functor $\widehat{\mathcal{C}_6} \leftarrow \widehat{\mathcal{C}_5}$. To subdivide a hexagon set X, choose $\overline{X}$ in the preimage of X in $\widehat{\overline{\mathcal{C}_6}}$ and apply the left adjoint $\widehat{\overline{\mathcal{C}_6}} \to \widehat{\mathcal{C}_5}$.

References

- "Categorical Tiling Theory: Constructing Directed Planar Tilings via Edge Reversal" - DiLeo, Sessoms, S. arXiv:2509.06363
- "Categorical Tiling Theory II: Schläfli Categories and Parametric Subdivisions" - Huffman, S. Work in progress

References

- "Categorical Tiling Theory: Constructing Directed Planar Tilings via Edge Reversal" - DiLeo, Sessoms, S. arXiv:2509.06363
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Thanks for coming!

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https://en.wikipedia.org/wiki/Order-5_square_tiling
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