Finite posets as algebraic expressions in duoidal categories

Brandon T. Shapiro* and David I. Spivak

University of Virginia*

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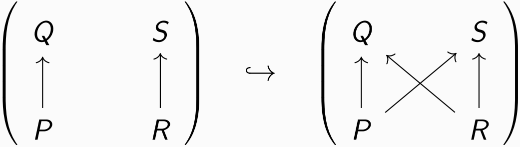
- 

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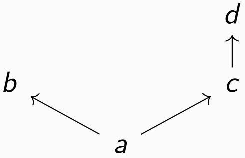

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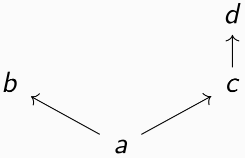

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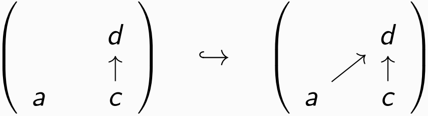

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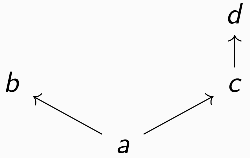

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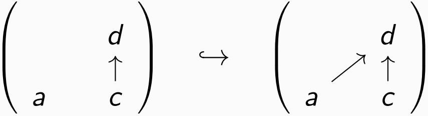

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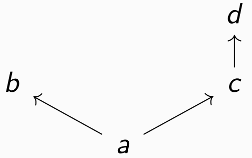

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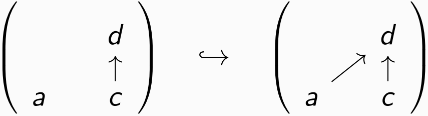

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A categorical operad for physical duoidal categories

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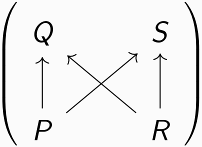

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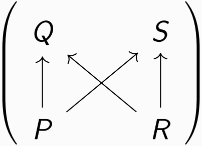

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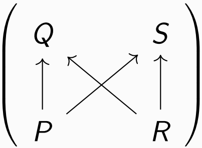

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(S.–Spivak) **Expr**-algebras are precisely physical duoidal categories.

Nonnegative reals and parallel programs

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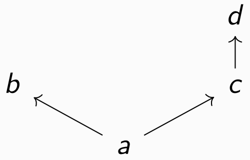
Given a finite set of programs and a poset of dependencies between them, the corresponding operation in $\mathbb{R}_{\geq 0}$ computes its optimal runtime.



References

- Brandon T. Shapiro and David I. Spivak, “Duoidal Structures for Compositional Dependence” arXiv:2210.01962

Thanks!