Finite posets as algebraic expressions in duoidal categories

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Duoidal categories

A *duoidal* category $\mathcal{C}$ consists of

- two monoidal structures $(I, \otimes)$ and $(J, \triangleleft)$;
- lax monoidal structures on the functors $\triangleleft\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and $J\colon 1 \to \mathcal{C}$ with respect to $(I, \otimes)$ (+ properties)
- equivalently, natural interchange morphisms (+ properties)

$$(a \triangleleft b) \otimes (c \triangleleft d) \to (a \otimes c) \triangleleft (b \otimes d)$$

$$I \to I \triangleleft I \qquad J \otimes J \to J \qquad I \to J$$

A *physical* duoidal category moreover has

- $\otimes$ is symmetric (compatibly with interchange)
- $I \cong J$ (compatibly with interchange)

Posets

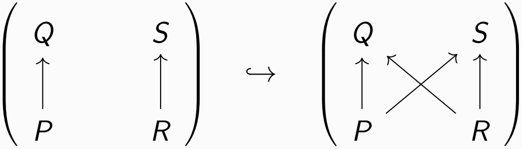
$$\left[\, (P \triangleleft Q) \otimes (R \triangleleft S) \to (P \otimes R) \triangleleft (Q \otimes S), \qquad \otimes \text{ symmetric}, \qquad I \cong J \,\right]$$

The category of posets has a physical duoidal structure:

- The unit is the empty poset $\varnothing$
- $\otimes$ is disjoint union, $\triangleleft$ is *join*

  $$P \otimes Q = \left(\; P \qquad\qquad Q \; \right) \qquad\qquad P \triangleleft Q = \begin{pmatrix} Q \\ \uparrow \\ P \end{pmatrix}$$

  where $P \triangleleft Q$ has elements $P \sqcup Q$ and $p < q$ for $p \in P$, $q \in Q$

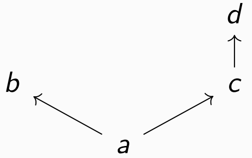
- 

Expressions

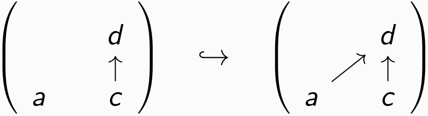
$$\left[\ (a \triangleleft b) \otimes (c \triangleleft d) \to (a \otimes c) \triangleleft (b \otimes d), \quad \otimes \text{ symmetric}, \quad I \cong J\ \right]$$

Physical duoidal categories have many compound operations and natural maps between them:

$$a \triangleleft (b \otimes (c \triangleleft d))$$



$$a \otimes (c \triangleleft d) \to (a \otimes c) \triangleleft d$$



(S.–Spivak) These operations and natural maps correspond to *expressible* posets and bijective monotone functions.

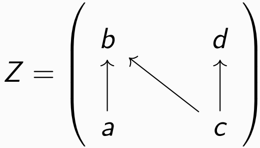
A poset is expressible if it is either empty or can be constructed from singleton posets by finitely many disjoint unions and joins.

Expressible posets

$$\left[\ (a \lhd b) \otimes (c \lhd d) \to (a \otimes c) \lhd (b \otimes d), \quad \otimes \text{ symmetric}, \quad I \cong J\ \right]$$

A poset is expressible if it is either empty or can be constructed from singleton posets by finitely many disjoint unions and joins.

Not all finite posets are expressible:



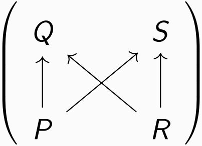
(S.–Spivak) A poset is expressible if and only if it is finite and admits no full embeddings of Z.

A categorical operad for physical duoidal categories

$$\left[\ (a \triangleleft b) \otimes (c \triangleleft d) \to (a \otimes c) \triangleleft (b \otimes d), \quad \otimes \text{ symmetric}, \quad I \cong J\ \right]$$

There is a categorical symmetric operad **Expr** where $\mathbf{Expr}_n$ is the category of expressible posets with n elements and bijective monotone functions. The unit is the singleton poset and operadic composition is given by "substitution"

$$\Big((a \otimes c) \triangleleft (b \otimes d)\Big); P, Q, R, S \qquad \mapsto$$



An **Expr**-algebra is a category $\mathcal{C}$ with coherent functors

$$\mathbf{Expr}_n \times \mathcal{C}^n \to \mathcal{C}.$$

(S.–Spivak) **Expr**-algebras are precisely physical duoidal categories.

Nonnegative reals and parallel programs

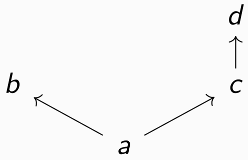
$$\left[(a \lhd b) \otimes (c \lhd d) \to (a \otimes c) \lhd (b \otimes d), \quad \otimes \text{ symmetric}, \quad I \cong J \right]$$

The posetal category $\mathbb{R}_{\geq 0}$ of nonnegative real numbers has a physical duoidal structure with unit 0, $\otimes$ given by maximum and $\lhd$ given by addition.

$$(a + b) \max (c + d) \leq (a \max c) + (b \max d)$$

max and $+$ correspond to how the runtimes of two programs behave when they are run in parallel and series, respectively.

Given a finite set of programs and a poset of dependencies between them, the corresponding operation in $\mathbb{R}_{\geq 0}$ computes its optimal runtime.



References

- Brandon T. Shapiro and David I. Spivak, “Duoidal Structures for Compositional Dependence” arXiv:2210.01962

Thanks!