Dynamic Operads for Evolving Organizations

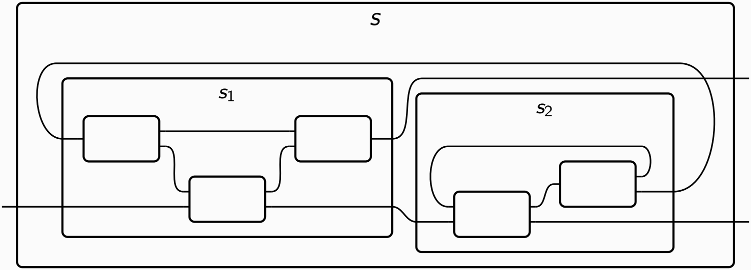
Brandon T. Shapiro* and David I. Spivak

JMM 2023

TOPOS INSTITUTE

Outline

1. Nested dynamic organizations



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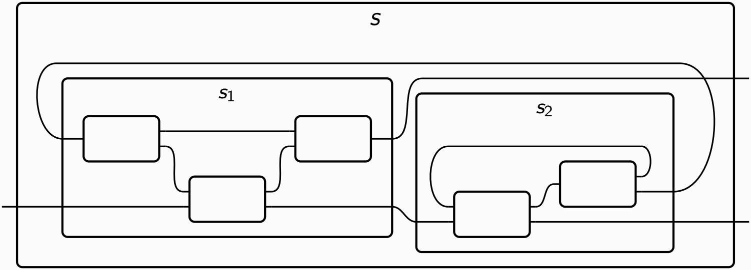

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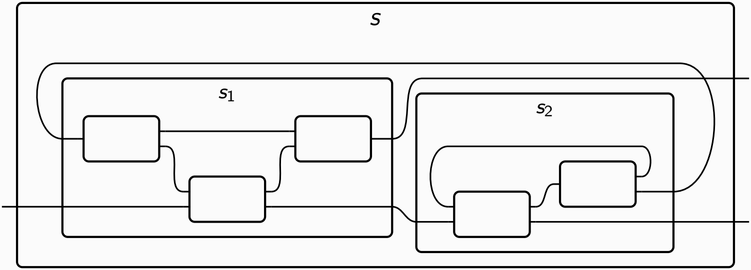

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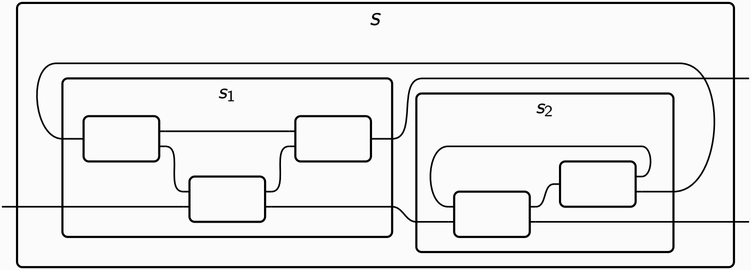

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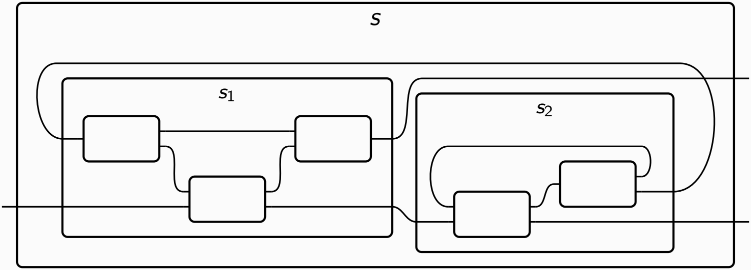

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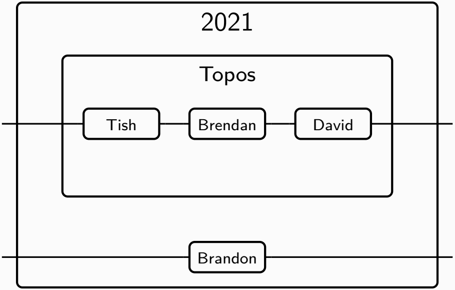

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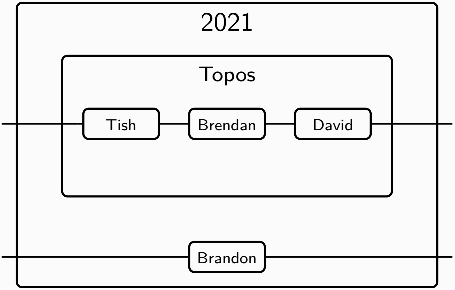

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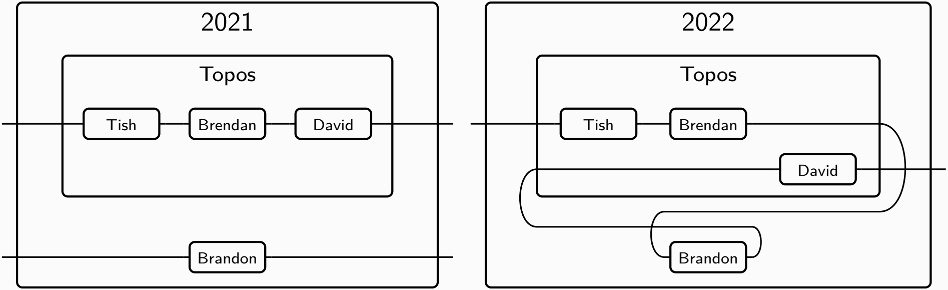

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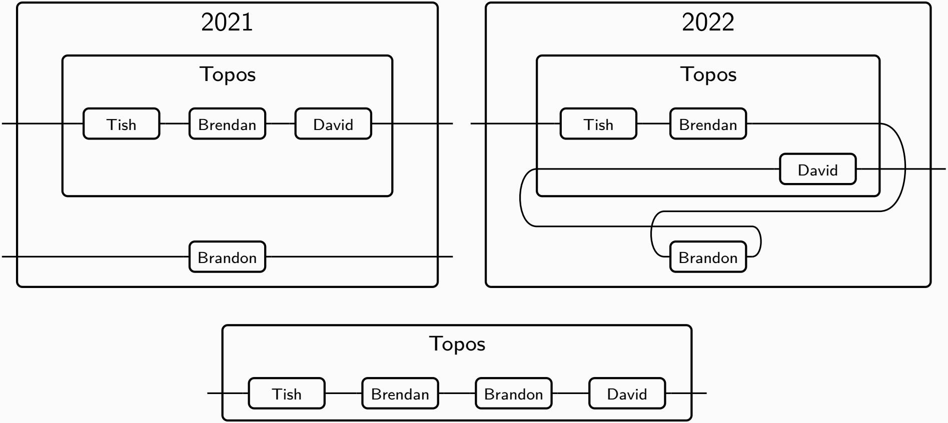

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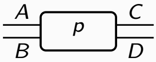

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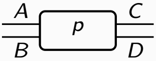

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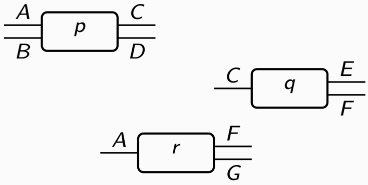

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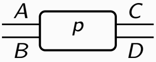

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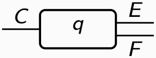

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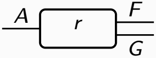

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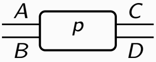

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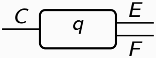

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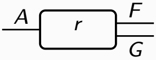

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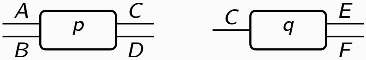

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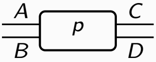

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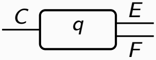

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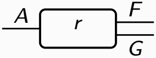

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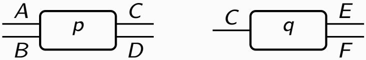

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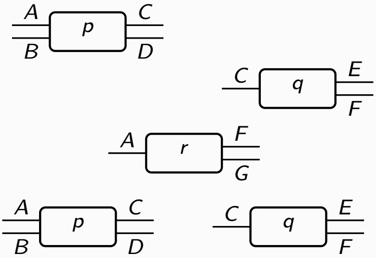

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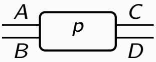

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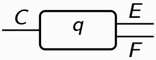

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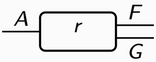

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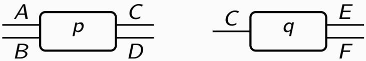

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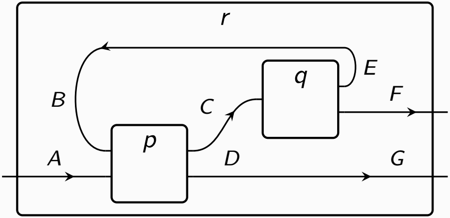

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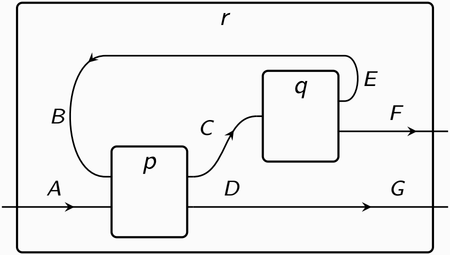

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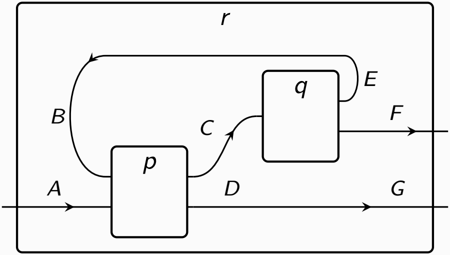

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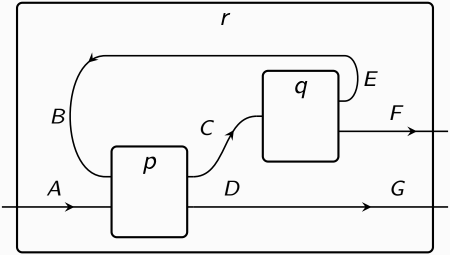

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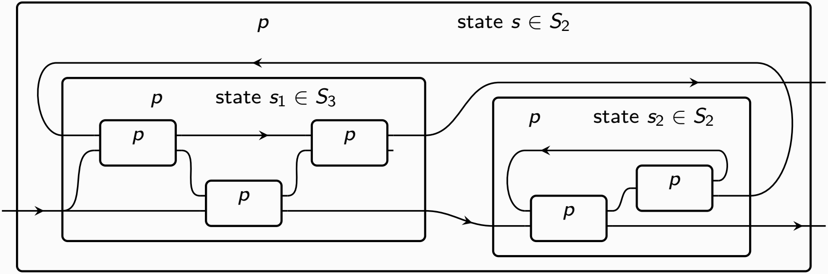

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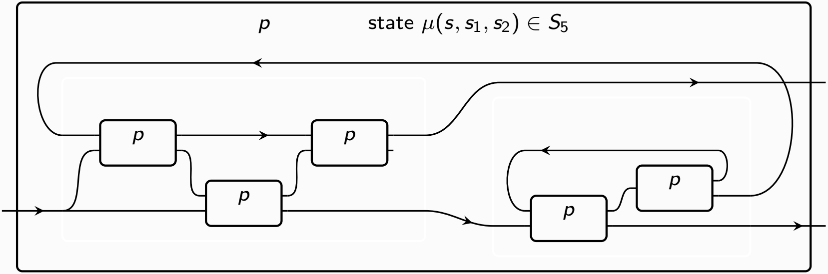

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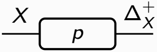

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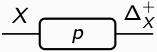

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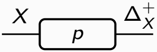

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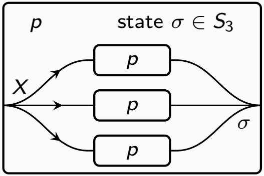

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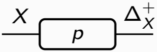

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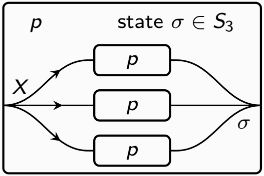
- The rewiring $(\Delta^+_X)^n \times X \to \Delta^+_n$ sends $\tau^1, ..., \tau^n, x$ to σ' where

$$\sigma'_i = \frac{\sigma_i \tau^i_x}{\sum_j \sigma_j \tau^j_x}$$



References

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Thanks!