Compositional Structure of Partial Evaluations

Carmen Constantin Tobias Fritz Paolo Perrone
Brandon Shapiro

Categories and Companions 6/11/21

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$1+2+3$

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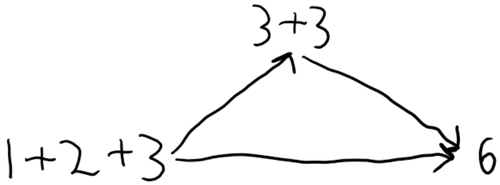

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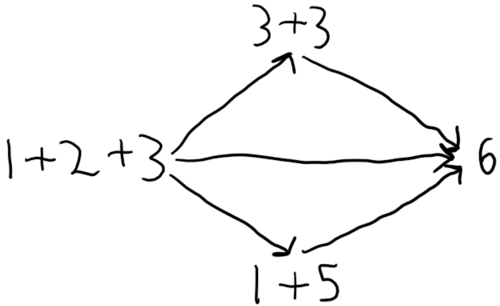

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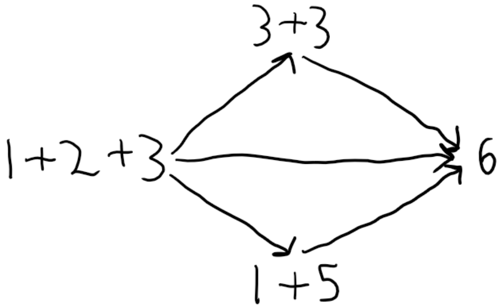

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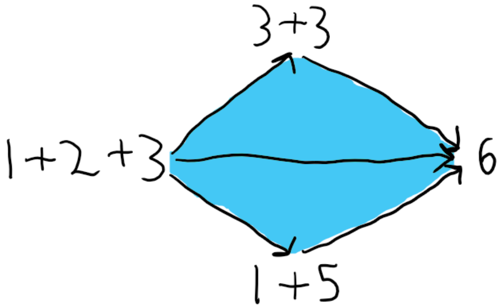

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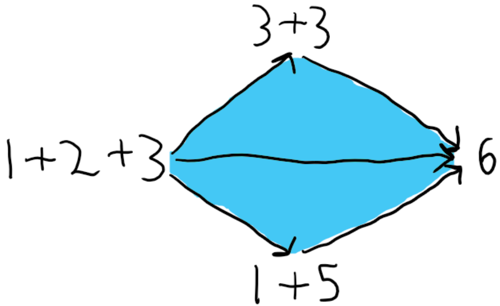
- Expressions can also be *partially evaluated*
- Partial evaluations form the paths in a simplicial set of nested formal expressions
- Do partial evaluations form the morphisms of a category?



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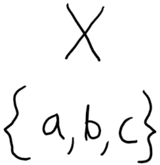

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$\{a, b, c\}$

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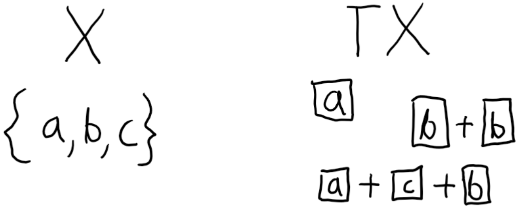

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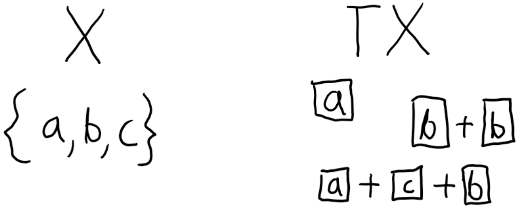

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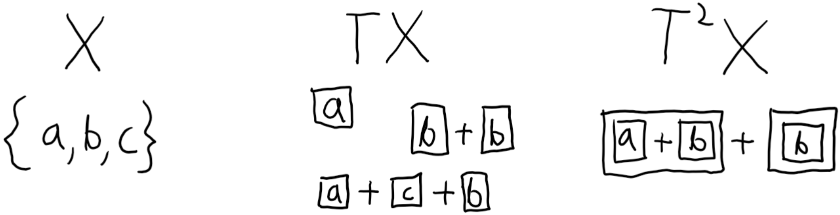

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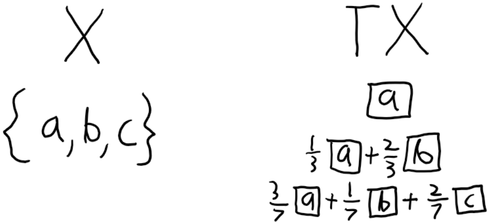

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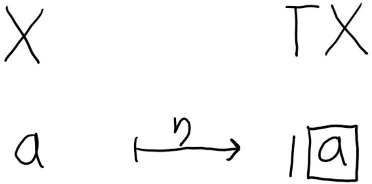

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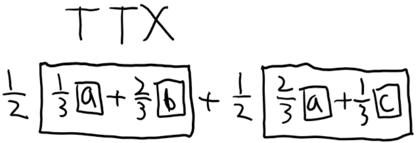

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$$\underbrace{\tfrac{1}{2}\boxed{\tfrac{1}{3}\boxed{a}+\tfrac{2}{3}\boxed{b}}+\tfrac{1}{2}\boxed{\tfrac{2}{3}\boxed{a}+\tfrac{1}{3}\boxed{c}}}_{TTX} \overset{\mu}{\longmapsto} \underbrace{\tfrac{1}{2}\boxed{a}+\tfrac{1}{3}\boxed{b}+\tfrac{1}{6}\boxed{c}}_{TX}$$

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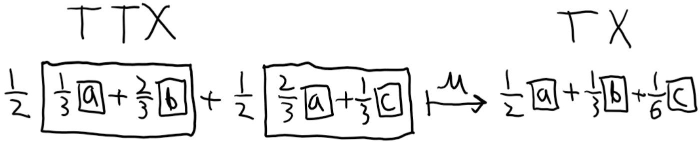

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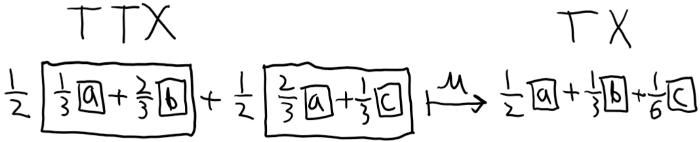

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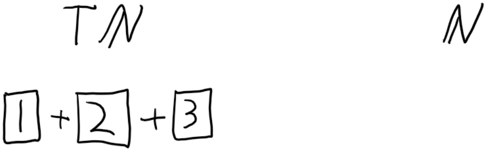

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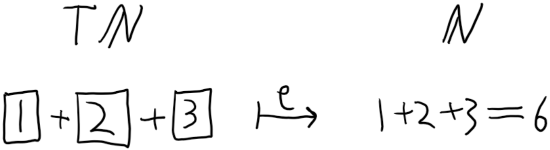

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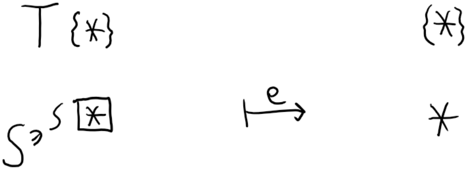

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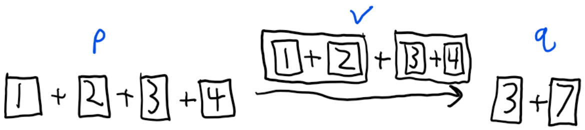

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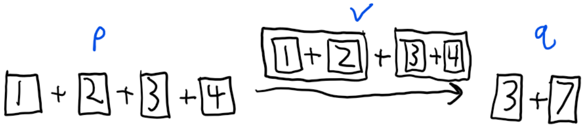

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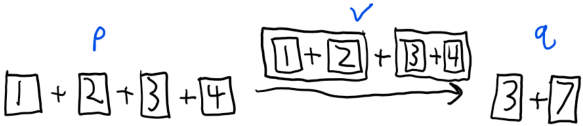

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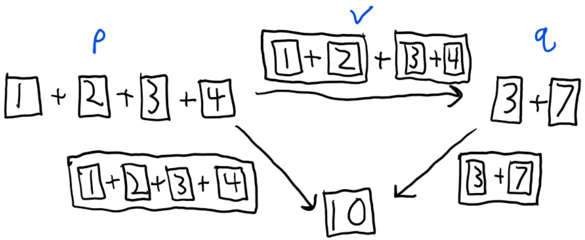

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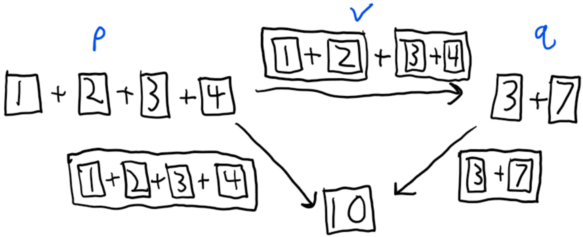

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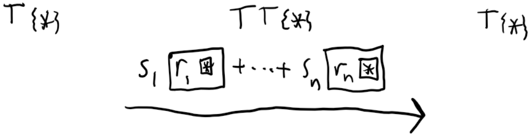

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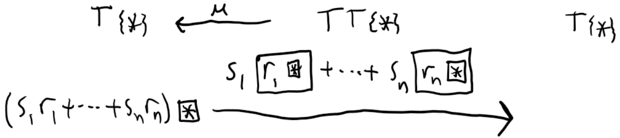

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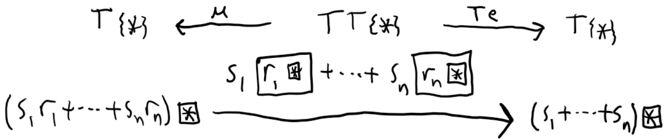

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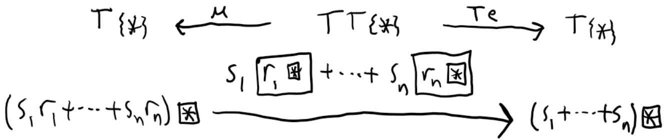

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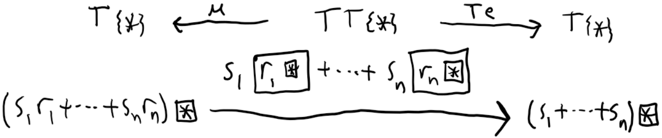

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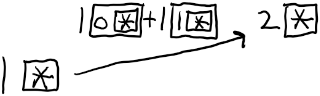

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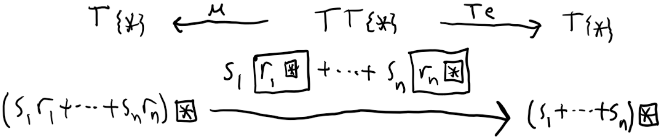

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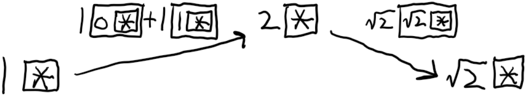

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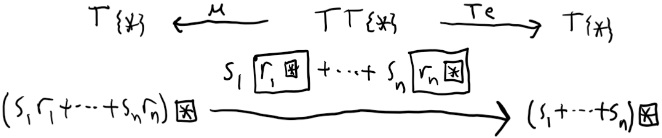

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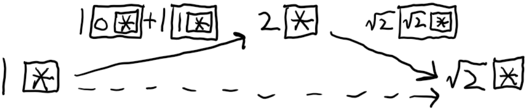

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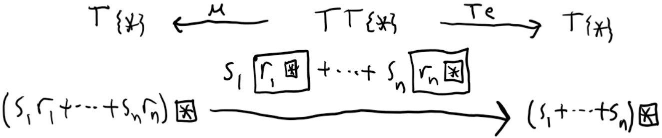

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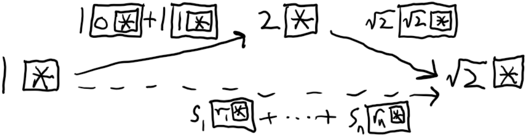

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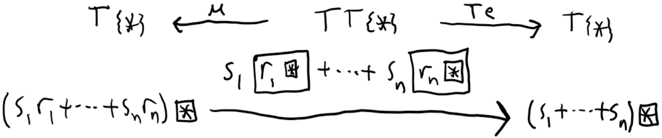

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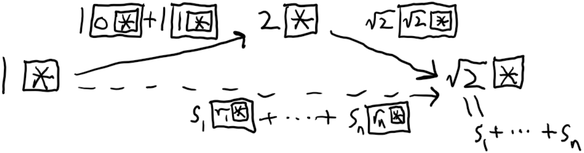

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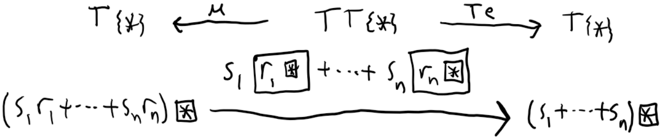

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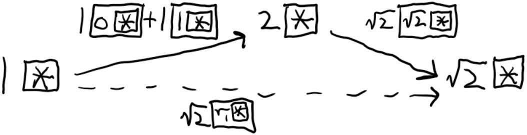

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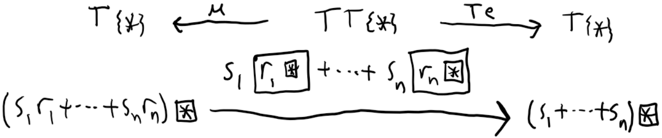

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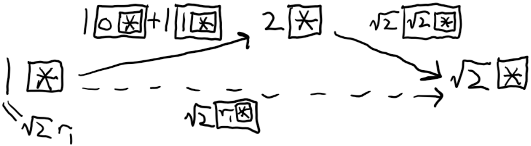

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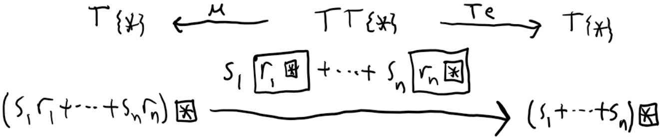

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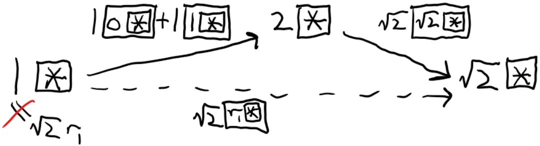

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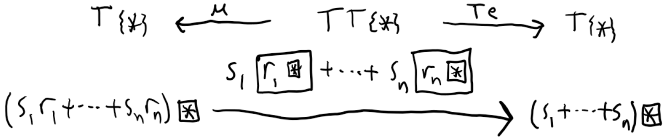

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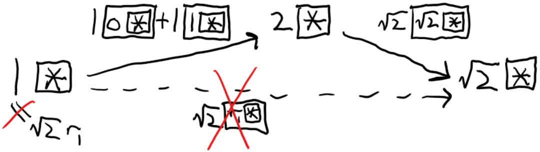

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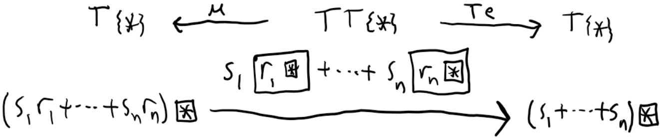

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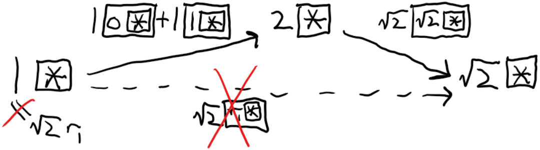

- (CFPS) Partial evaluations don't always compose

Bar Construction

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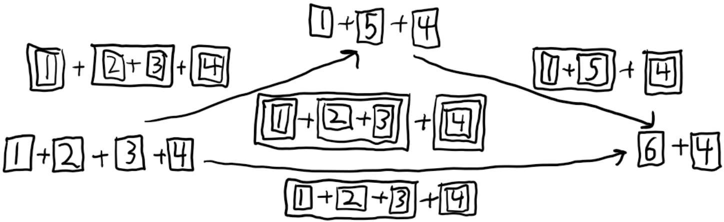

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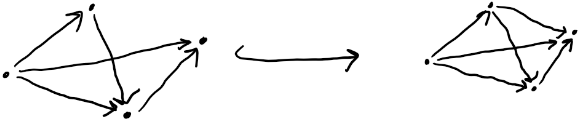

$$\begin{array}{ccc} \Delta^{n-1} \sqcup_{\Delta^{n-2}} \Delta^{n-1} & \xrightarrow{\forall} & X \\ \big\downarrow & \nearrow_{\exists} & \\ \Delta^n & & \end{array}$$

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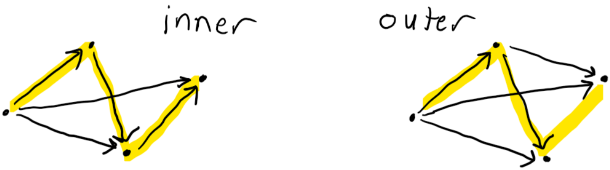

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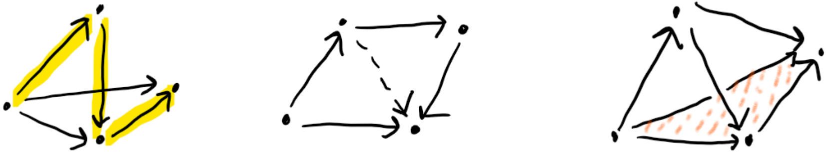

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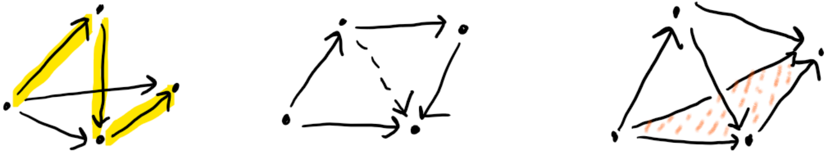

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Databases

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- A data table

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A	B	C	D
0	1	0	0
0	2	3	4
5	1	3	0

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- A data table can be split into subtables on fewer attributes

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5	1

B	C
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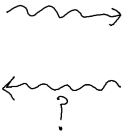

A	B
0	1
0	2
5	1

B	C
1	0
2	3
1	3

A	C	D
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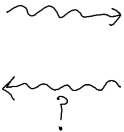

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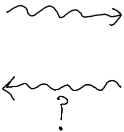

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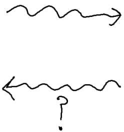
- A table can only be reliably recovered from an (undirected) acyclic configuration of subtables

A	B	C	D
0	1	0	0
0	2	3	4
5	1	3	0



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Thank you!

- Carmen Constantin, Tobias Fritz, Paolo Perrone, and Brandon Shapiro. Partial evaluations and the compositional structure of the bar construction. arXiv:2009.07302.
- Carmen Constantin, Tobias Fritz, Paolo Perrone, and Brandon Shapiro. Weak cartesian properties of simplicial sets. arXiv:2105.04775.
- Catriel Beeri, Ronald Fagin, David Maier, and Mihalis Yannakakis. On the Desirability of Acyclic Database Schemes. *Journal of the ACM*, 30, 479-513, 1983.