

Compositional Structure of Partial Evaluations

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Categories and Companions 6/11/21

Compositional Structure of Partial Evaluations

- Algebra is all about evaluating *formal expressions*

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$$1+2+3$$

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$$1 + 2 + 3 \longrightarrow 6$$

Compositional Structure of Partial Evaluations

- Algebra is all about evaluating *formal expressions*
- Expressions can also be *partially evaluated*

$$1+2+3 \longrightarrow 6$$

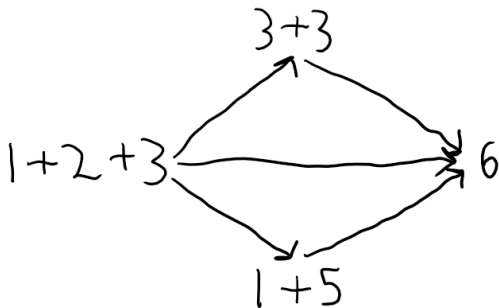
Compositional Structure of Partial Evaluations

- Algebra is all about evaluating *formal expressions*
- Expressions can also be *partially evaluated*



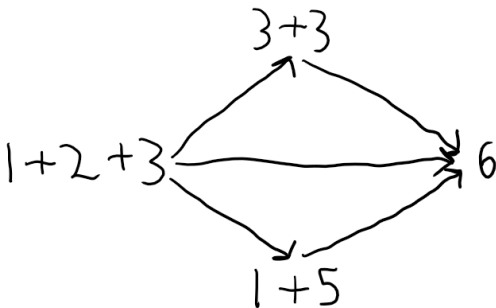
Compositional Structure of Partial Evaluations

- Algebra is all about evaluating *formal expressions*
- Expressions can also be *partially evaluated*



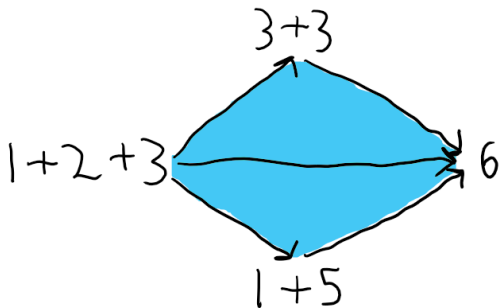
Compositional Structure of Partial Evaluations

- Algebra is all about evaluating *formal expressions*
- Expressions can also be *partially evaluated*
- Partial evaluations form the paths in a simplicial set of nested formal expressions



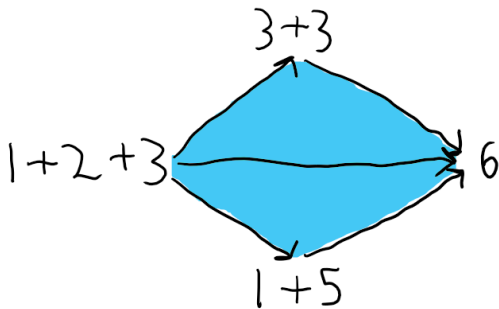
Compositional Structure of Partial Evaluations

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Compositional Structure of Partial Evaluations

- Algebra is all about evaluating *formal expressions*
- Expressions can also be *partially evaluated*
- Partial evaluations form the paths in a simplicial set of nested formal expressions
- Do partial evaluations form the morphisms of a category?



Monads

For T a monad on $\mathcal{S}et$ and X a set:

Monads

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X
 $\{a, b, c\}$

Monads

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- Elements of TX are *formal expressions* on X

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Example: “Free (commutative) monoid” monad

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Example: “Free (commutative) monoid” monad

$$\begin{array}{l} X \\ \{a, b, c\} \end{array} \qquad \begin{array}{l} TX \\ \boxed{a} \quad \boxed{b} + \boxed{b} \\ \boxed{a} + \boxed{c} + \boxed{b} \end{array}$$

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For T a monad on $\mathcal{S}et$ and X a set:

- Elements of TX are *formal expressions* on X
- Elements of $T^n X$ are *nested formal expressions*

Example: “Free (commutative) monoid” monad

X
 $\{a, b, c\}$

TX
 \boxed{a} $\boxed{b} + \boxed{b}$
 $\boxed{a} + \boxed{c} + \boxed{b}$

Monads

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Example: “Free (commutative) monoid” monad

$$X$$
$$\{a, b, c\}$$

$$TX$$
$$\boxed{a} \quad \boxed{b} + \boxed{b}$$
$$\boxed{a} + \boxed{c} + \boxed{b}$$

$$T^2 X$$
$$\boxed{\boxed{a} + \boxed{b}} + \boxed{\boxed{b}}$$

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Example: Distribution monad

$$X \\ \{a, b, c\}$$

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Example: Distribution monad

$$X$$
$$\{a, b, c\}$$

$$TX$$
$$\boxed{a}$$
$$\frac{1}{3}\boxed{a} + \frac{2}{3}\boxed{b}$$
$$\frac{3}{7}\boxed{a} + \frac{1}{7}\boxed{b} + \frac{2}{7}\boxed{c}$$

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Example: Distribution monad

$$\begin{array}{ccc} X & & TX \\ a & \xrightarrow{\eta} & |a| \end{array}$$

For T a monad on Set and X a set:

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Example: Distribution monad

$$TTX$$
$$\frac{1}{2} \left[\frac{1}{3} a + \frac{2}{3} b \right] + \frac{1}{2} \left[\frac{2}{3} a + \frac{1}{3} c \right]$$

Monads

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Example: Distribution monad

$$\begin{array}{ccc} TTX & & TX \\ \frac{1}{2} \boxed{\frac{1}{3} a + \frac{2}{3} b} + \frac{1}{2} \boxed{\frac{2}{3} a + \frac{1}{3} c} & \xrightarrow{\mu} & \frac{1}{2} a + \frac{1}{3} b + \frac{1}{6} c \end{array}$$

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For T a monad on Set and X a set:

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Example: Free S -module monad (S a semiring)

$$\begin{array}{ccc} TT X & & TX \\ \frac{1}{2} \boxed{\frac{1}{3} a + \frac{2}{3} b} + \frac{1}{2} \boxed{\frac{2}{3} a + \frac{1}{3} c} & \xrightarrow{\mu} & \frac{1}{2} a + \frac{1}{3} b + \frac{1}{6} c \end{array}$$

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$$e : TA \rightarrow A$$

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Example: (Commutative) monoid \mathbb{N}

$T\mathbb{N}$

\mathbb{N}

$$\boxed{1} + \boxed{2} + \boxed{3}$$

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Example: (Commutative) monoid \mathbb{N}

$$T\mathbb{N} \qquad \mathbb{N}$$
$$\boxed{1} + \boxed{2} + \boxed{3} \xrightarrow{e} 1 + 2 + 3 = 6$$

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Example: Trivial S -module

$$T \{*\}$$
$$\{*\}$$
$$S \boxed{*}$$

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Partial evaluations

- Consider a T -algebra (A, e)

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Example: (Commutative) monoid \mathbb{N}

$$\boxed{1} + \boxed{2} + \boxed{3} + \boxed{4}$$

$$\boxed{3} + \boxed{7}$$

Partial evaluations

- Consider a T -algebra (A, e)
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Example: (Commutative) monoid \mathbb{N}

$$\boxed{1} + \boxed{2} + \boxed{3} + \boxed{4}$$

ρ

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ρ

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$$\boxed{1} + \boxed{2} + \boxed{3} + \boxed{4} \xrightarrow{\quad} \boxed{\boxed{1+2} + \boxed{3+4}} \xrightarrow{\quad} \boxed{3} + \boxed{7}$$

Partial evaluations

- Consider a T -algebra (A, e)
- A *partial evaluation* is a doubly nested expression $v \in TTA$
- The source p of v is $\mu(v)$ (remove outer boxes)

Example: (Commutative) monoid \mathbb{N}

The diagram illustrates a partial evaluation process in a monoid. It starts with the expression $1 + 2 + 3 + 4$, where the source p is indicated by a blue letter above the expression. An arrow points to a doubly nested expression $(1 + 2) + (3 + 4)$, where the partial evaluation v is indicated by a blue letter above the expression. A second arrow points to the result $3 + 7$, where the source q is indicated by a blue letter above the expression.

Partial evaluations

- Consider a T -algebra (A, e)
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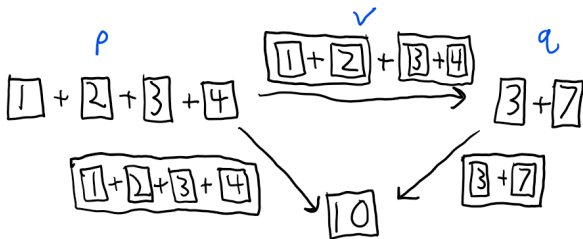
Example: (Commutative) monoid \mathbb{N}

$$\boxed{1} + \boxed{2} + \boxed{3} + \boxed{4} \xrightarrow{\boxed{1+2} + \boxed{3+4}} \boxed{3} + \boxed{7}$$

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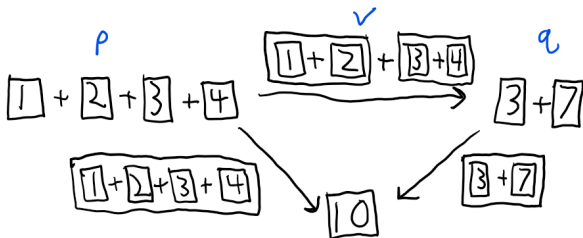
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- Do partial evaluations compose?

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- Consider the trivial S -module:

$$\begin{array}{ccc} T\{*\} & & T\{*\} \\ & TT\{*\} & \\ & s_1 \boxed{r_1} + \dots + s_n \boxed{r_n} & \\ & \xrightarrow{\hspace{10em}} & \end{array}$$

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 (s_1 r_1 + \dots + s_n r_n) \boxtimes & \xrightarrow{\quad\quad\quad} & & &
 \end{array}$$

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- Let $S = \mathbb{N}[\sqrt{2}] = \{n + m\sqrt{2}\}$

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$$\begin{array}{ccc} & \boxed{0} + \boxed{1} & \rightarrow \quad \boxed{2} \\ & \uparrow & \\ \boxed{1} & & \end{array}$$

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 (s_1 r_1 + \dots + s_n r_n) * & \xrightarrow{\quad\quad\quad} & & & (s_1 + \dots + s_n) *
 \end{array}$$

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$$\begin{array}{ccc}
 & \boxed{0 * } + \boxed{1 * } & \rightarrow & \boxed{2 * } & \xrightarrow{\sqrt{2} \boxed{\sqrt{2} * }} & \boxed{\sqrt{2} * } \\
 \boxed{1 * } & \nearrow & & & \searrow & \\
 & & & & &
 \end{array}$$

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$$\begin{array}{ccc}
 & \boxed{0 \boxtimes} + \boxed{1 \boxtimes} & \rightarrow & \boxed{2 \boxtimes} & \xrightarrow{\sqrt{2} \boxtimes} & \boxed{\sqrt{2} \boxtimes} \\
 \boxed{1 \boxtimes} & \nearrow & & & \searrow & \\
 & \text{-----} & & & \text{-----} & \rightarrow & \boxed{\sqrt{2} \boxtimes}
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$$\begin{array}{ccc}
 | \boxed{\ast} & \xrightarrow{\quad\quad\quad} & | \boxed{0 \boxtimes} + | \boxed{1 \boxtimes} & \xrightarrow{\quad\quad\quad} & 2 \boxed{\ast} & \xrightarrow{\quad\quad\quad} & \sqrt{2} \boxed{\sqrt{2} \boxtimes} \\
 & \xrightarrow{\quad\quad\quad} & & & & \searrow & \sqrt{2} \boxed{\ast} \\
 & \xrightarrow{\quad\quad\quad} & & & & \xrightarrow{\quad\quad\quad} & \\
 & \xrightarrow{\quad\quad\quad} & s_1 \boxed{r_1 \boxtimes} + \dots + s_n \boxed{r_n \boxtimes} & & & &
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 \boxed{1 \ast} & \nearrow & & & \searrow & \\
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- (CFPS) Partial evaluations don't always compose

Bar Construction

- Partial evaluations form a simplicial set, called the *Bar Construction* of a T -algebra A

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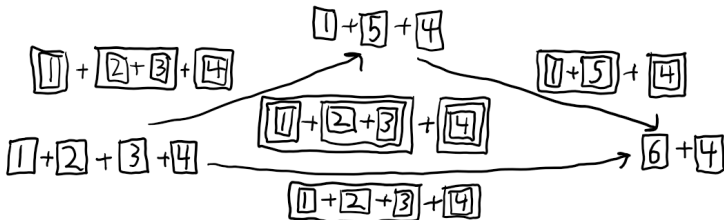
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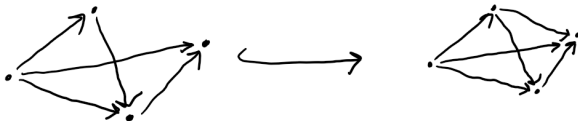
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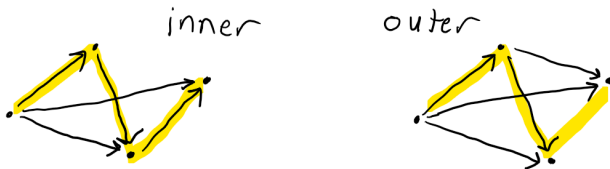
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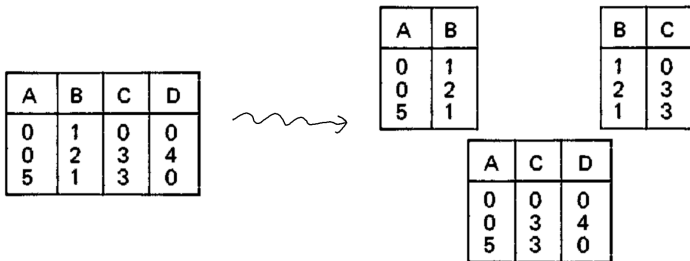
A	B	C	D
0	1	0	0
0	2	3	4
5	1	3	0

- A data table can be split into subtables on fewer attributes

A	B	C	D
0	1	0	0
0	2	3	4
5	1	3	0

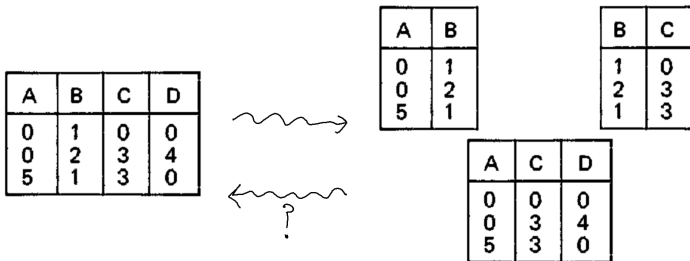
Databases

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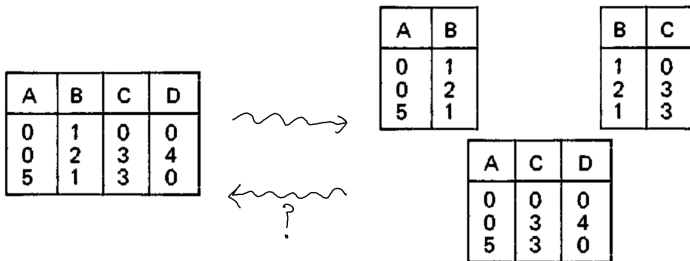
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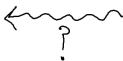
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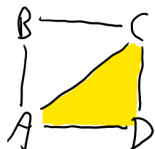


A	B
0	1
0	2
5	1

B	C
1	0
2	3
1	3



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0	0	0
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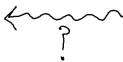
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0	1	0	0
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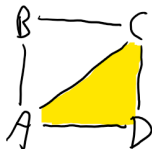


A	B
0	1
0	2
5	1

B	C
1	0
2	3
1	3



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Thank you!

- Carmen Constantin, Tobias Fritz, Paolo Perrone, and Brandon Shapiro. Partial evaluations and the compositional structure of the bar construction. [arXiv:2009.07302](https://arxiv.org/abs/2009.07302).
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- Catriel Beeri, Ronald Fagin, David Maier, and Mihalis Yannakakis. On the Desirability of Acyclic Database Schemes. *Journal of the ACM*, 30, 479-513, 1983.