Compositional Structure of Partial Evaluations

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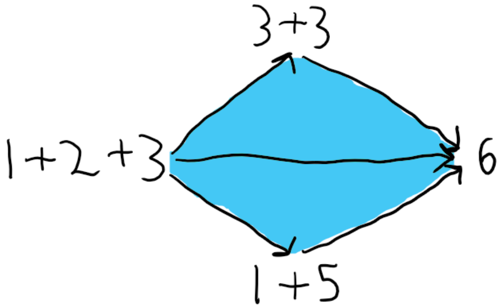
Categories and Companions 6/11/21

Compositional Structure of Partial Evaluations

- Algebra is all about evaluating *formal expressions*
- Expressions can also be *partially evaluated*
- Partial evaluations form the paths in a simplicial set of nested formal expressions
- Do partial evaluations form the morphisms of a category?



Monads

For T a monad on $\mathcal{S}et$ and X a set:

- Elements of TX are *formal expressions* on X
- Elements of T^nX are *nested* formal expressions

Example: "Free (commutative) monoid" monad

X	TX	T^2X
$\{a, b, c\}$	$\boxed{a}$ $\boxed{b}+\boxed{b}$ $\boxed{a}+\boxed{c}+\boxed{b}$	$\boxed{\boxed{a}+\boxed{b}}+\boxed{\boxed{b}}$

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Example: Distribution monad

X

$\{a, b, c\}$

TX

$\boxed{a}$

$\frac{1}{3}\boxed{a} + \frac{2}{3}\boxed{b}$

$\frac{3}{7}\boxed{a} + \frac{1}{7}\boxed{b} + \frac{2}{7}\boxed{c}$

For T a monad on $\mathcal{S}et$ and X a set:

- Elements of TX are *formal expressions* on X
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Example: Distribution monad

$$\begin{array}{ccc} X & & TX \\ a & \xmapsto{\eta} & 1\boxed{a} \end{array}$$

Monads

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Example: Distribution monad

$$\underbrace{\tfrac{1}{2}\boxed{\tfrac{1}{3}\boxed{a}+\tfrac{2}{3}\boxed{b}}+\tfrac{1}{2}\boxed{\tfrac{2}{3}\boxed{a}+\tfrac{1}{3}\boxed{c}}}_{TTX} \xmapsto{\mu} \underbrace{\tfrac{1}{2}\boxed{a}+\tfrac{1}{3}\boxed{b}+\tfrac{1}{6}\boxed{c}}_{TX}$$

Monads

For T a monad on $\mathcal{S}et$ and X a set:

- Elements of TX are *formal expressions* on X
- Elements of T^nX are *nested* formal expressions
- An algebra A of T is equipped with an *evaluation map*

$$e : TA \to A$$

Example: Free S-module monad (S a semiring)

$$\underset{TTX}{\underbrace{\frac{1}{2}\boxed{\frac{1}{3}\boxed{a}+\frac{2}{3}\boxed{b}}+\frac{1}{2}\boxed{\frac{2}{3}\boxed{a}+\frac{1}{3}\boxed{c}}}} \xmapsto{\mu} \underset{TX}{\underbrace{\frac{1}{2}\boxed{a}+\frac{1}{3}\boxed{b}+\frac{1}{6}\boxed{c}}}$$

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Example: (Commutative) monoid $\mathbb{N}$

$$T\mathbb{N} \qquad\qquad\qquad \mathbb{N}$$

$$\boxed{1}+\boxed{2}+\boxed{3} \xmapsto{e} 1+2+3=6$$

Monads

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Example: Trivial S-module

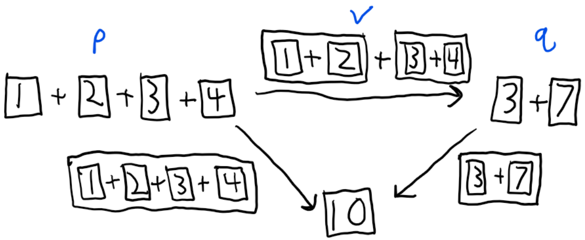
$$T\{*\} \qquad\qquad \{*\}$$

$$S \ni s\,\boxed{*} \quad \xmapsto{e} \quad *$$

Partial evaluations

- Consider a T-algebra (A, e)
- A *partial evaluation* is a doubly nested expression $v \in TTA$
- The source p of v is $\mu(v)$ (remove outer boxes)
- The target q of v is $Te(v)$ (remove inner boxes)

Example: (Commutative) monoid $\mathbb{N}$



- Do partial evaluations compose?

Do Partial Evaluations Compose?

- A *partial evaluation* from p to q is a doubly nested expression $v \in TTA$ with $\mu(v) = p$ and $Te(v) = q$
- Consider the trivial S-module:

$$
\begin{tikzcd}
T\{*\} & TT\{*\} \arrow[l, "\mu"'] \arrow[r, "Te"] & T\{*\}
\end{tikzcd}
$$

$$
(s_1 r_1 + \cdots + s_n r_n)\boxed{*} \xrightarrow{\;s_1 \boxed{r_1 \boxed{*}} + \cdots + s_n \boxed{r_n \boxed{*}}\;} (s_1 + \cdots + s_n)\boxed{*}
$$

- Let $S = \mathbb{N}[\sqrt{2}] = \{n + m\sqrt{2}\}$

$$
\begin{tikzcd}
 & 2\boxed{*} \arrow[dr, "\sqrt{2}\boxed{\sqrt{2}\boxed{*}}"] & \\
1\boxed{*} \arrow[ur, "1\boxed{0\boxed{*}} + 1\boxed{1\boxed{*}}"] \arrow[rr, dashed, "s_1\boxed{r_1\boxed{*}} + \cdots + s_n\boxed{r_n\boxed{*}}"'] & & \sqrt{2}\boxed{*} \arrow[d, phantom, "\shortparallel"] \\
 & & s_1 + \cdots + s_n
\end{tikzcd}
$$

Do Partial Evaluations Compose?

- A *partial evaluation* from p to q is a doubly nested expression $v \in TTA$ with $\mu(v) = p$ and $Te(v) = q$
- Consider the trivial S-module:

$$T\{*\} \xleftarrow{\mu} TT\{*\} \xrightarrow{Te} T\{*\}$$

$$(s_1 r_1 + \cdots + s_n r_n)\boxed{*} \xrightarrow{s_1\boxed{r_1\boxed{*}} + \cdots + s_n\boxed{r_n\boxed{*}}} (s_1 + \cdots + s_n)\boxed{*}$$

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$1 = \sqrt{2}\, r_1$

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- Consider the trivial S-module:

$$\begin{tikzcd} T\{*\} & TT\{*\} \arrow[l, "\mu"'] \arrow[r, "Te"] & T\{*\} \\ & s_1\boxed{r_1\boxed{*}}+\cdots+s_n\boxed{r_n\boxed{*}} & \\ (s_1r_1+\cdots+s_nr_n)\boxed{*} \arrow[rr] & & (s_1+\cdots+s_n)\boxed{*} \end{tikzcd}$$

- Let $S = \mathbb{N}[\sqrt{2}] = \{n + m\sqrt{2}\}$

$$\begin{tikzcd} 1\boxed{*} \arrow[r, "1\boxed{0\boxed{*}}+1\boxed{1\boxed{*}}"] \arrow[rr, dashed, bend right, "\cancel{\sqrt{2}\boxed{r_1\boxed{*}}}"'] & 2\boxed{*} \arrow[r, "\sqrt{2}\boxed{\sqrt{2}\boxed{*}}"] & \sqrt{2}\boxed{*} \end{tikzcd}$$

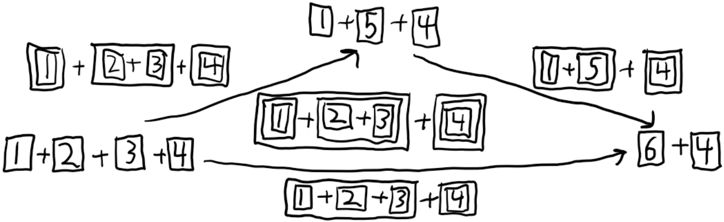
$1 \neq \sqrt{2}\, r_1$

- (CFPS) Partial evaluations don't always compose

Bar Construction

- Partial evaluations form a simplicial set, called the *Bar Construction* of a T-algebra A
- n-simplices of $Bar_T(A)$ are $(n+1)$-nested formal expressions

$$\cdots T^4A \;\begin{array}{c}\xrightarrow{T^3e}\\ \xrightarrow{T^2\mu}\\ \xrightarrow{T\mu}\\ \xrightarrow{\mu}\end{array}\; T^3A \;\begin{array}{c}\xrightarrow{T^2e}\\ \xrightarrow{T\mu}\\ \xrightarrow{\mu}\end{array}\; T^2A \;\begin{array}{c}\xrightarrow{Te}\\ \xrightarrow{\mu}\end{array}\; TA$$



Compositional Structure

- If T arises from a nonsymmetric operad (i.e. monoids, semigroups, M-sets), $Bar_T(A)$ is the nerve of a category
- If T arises from a symmetric operad (i.e. commutative monoids), $Bar_T(A)$ is generally not a category
- $Bar_{CommMon}(\mathbb{N})$ is not even a quasicategory (CFPS)
- In this case, $Bar_T(A)$ is *inner span complete* (CFPS)

$$\begin{tikzcd} \Delta^{n-1} \sqcup_{\Delta^{n-2}} \Delta^{n-1} \arrow[r, "\forall"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

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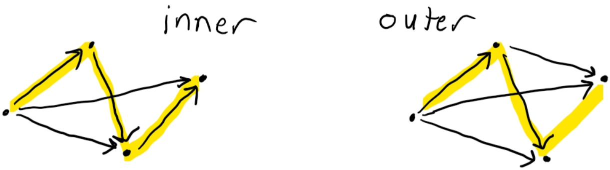

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Filler Conditions

- What other fillers do inner span complete simplicial sets have?
- (CFPS) All *directed acyclic configurations* $S \subset \Delta^n$:
  - S contains the spine of Δ^n (directedness)
  - The 1-skeleton of S is *chordal*
  - Any k-simplex boundary $\partial\Delta^k$ in S is filled by a k-simplex Δ^k
- Horns are not acyclic

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  - Any k-simplex boundary $\partial\Delta^k$ in S is filled by a k-simplex Δ^k
- Horns are not acyclic
- Spine inclusions are acyclic

Databases

- A data table can be split into subtables on fewer attributes
- Here (0,1,3,4) fits into the subtables, but not the entire table
- A table can only be reliably recovered from an (undirected) acyclic configuration of subtables

A	B	C	D
0	1	0	0
0	2	3	4
5	1	3	0

?

A	B
0	1
0	2
5	1

B	C
1	0
2	3
1	3

A	C	D
0	0	0
0	3	4
5	3	0

Thank you!

- Carmen Constantin, Tobias Fritz, Paolo Perrone, and Brandon Shapiro. Partial evaluations and the compositional structure of the bar construction. arXiv:2009.07302.
- Carmen Constantin, Tobias Fritz, Paolo Perrone, and Brandon Shapiro. Weak cartesian properties of simplicial sets. arXiv:2105.04775.
- Catriel Beeri, Ronald Fagin, David Maier, and Mihalis Yannakakis. On the Desirability of Acyclic Database Schemes. *Journal of the ACM*, 30, 479-513, 1983.