Dynamic PROPs for Networked Learners

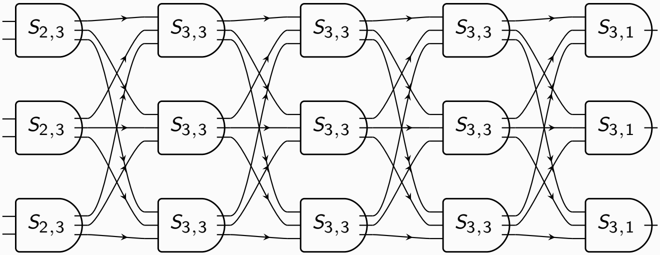
Brandon T. Shapiro* and David I. Spivak

CALCO 2023

TOPOS INSTITUTE

Outline

1. Nested dynamic structures
2. Polynomials and wiring diagrams
3. Dynamics: polynomial coalgebras
4. Nesting and networks: operad and PROP structure
5. A dynamic PROP for deep learning
6. Vistas



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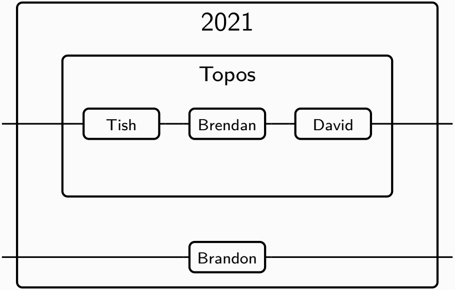

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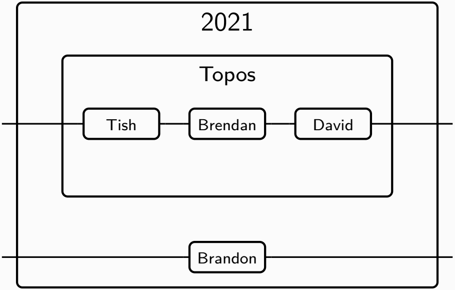

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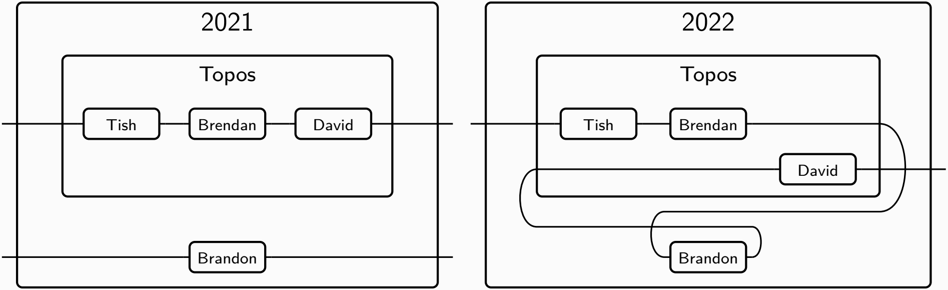

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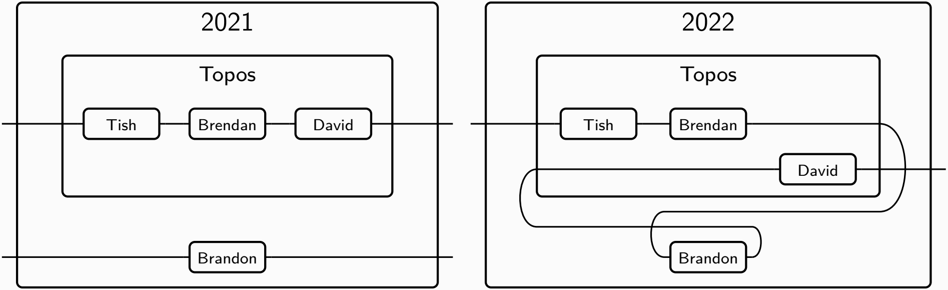

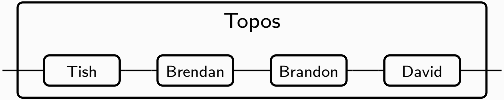

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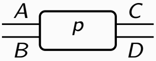

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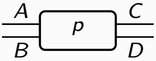

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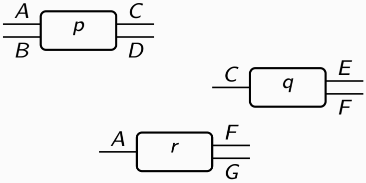

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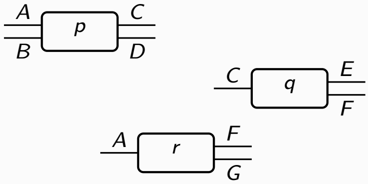

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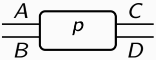

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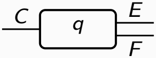

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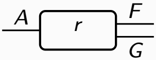

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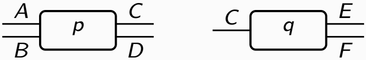

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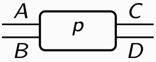

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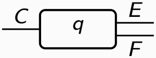

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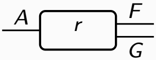

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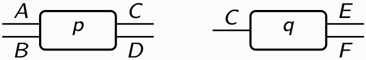

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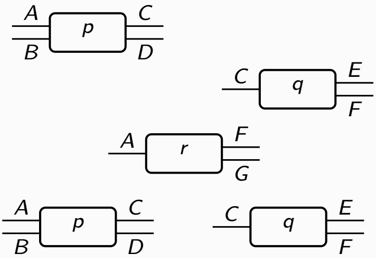

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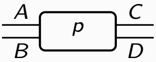

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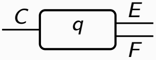

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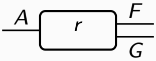

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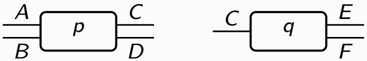

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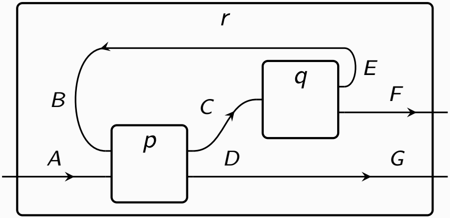

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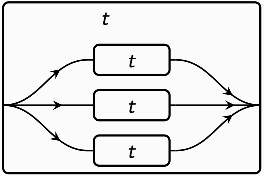

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- the feedback $x \in \mathbb{R}^n$ causes (k, f, r) to update to

$$(k, f, r + \epsilon \pi_1 Df^\top x)$$

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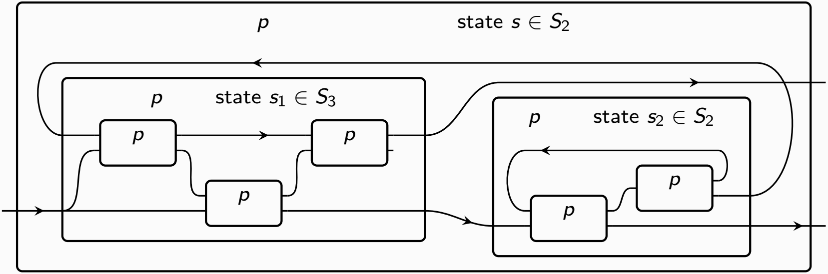

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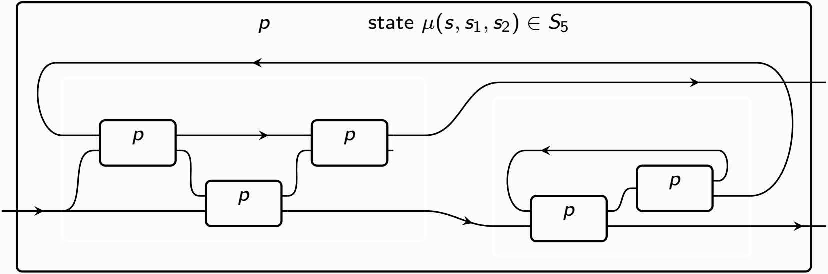

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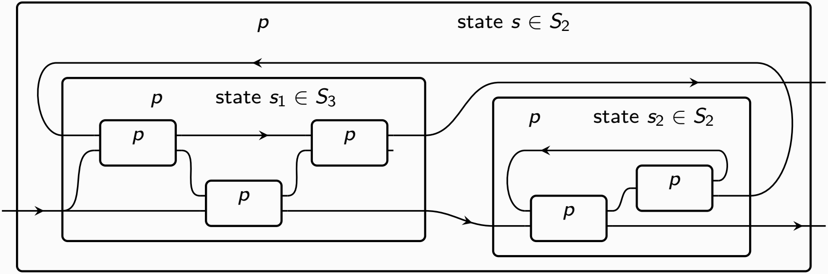

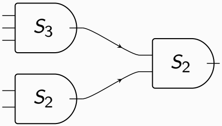

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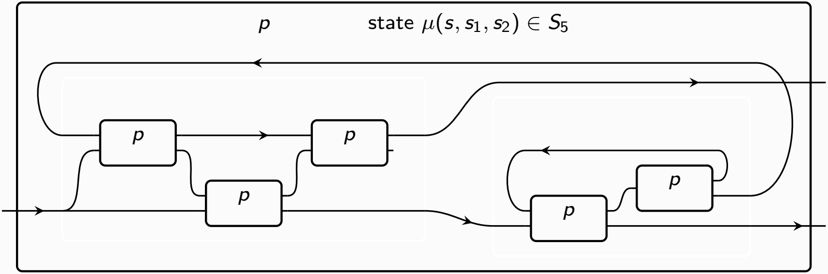

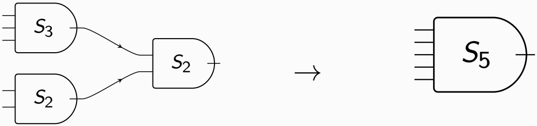

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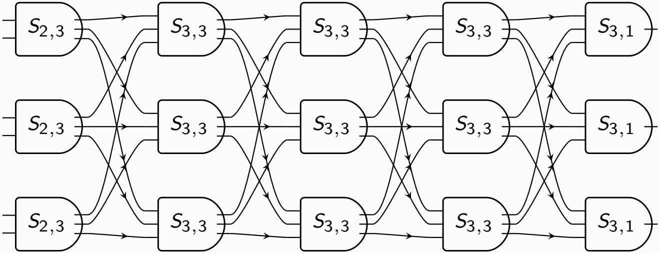

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- $(0, \mathrm{id}_{\mathbb{R}}, *)$, $(0, \mathrm{id}_*, *)$
- $(\sigma, \tau, (k, f, r)) \mapsto$
$$(k, \mathbb{R}^k \times \mathbb{R}^m \xrightarrow{\mathrm{id}\times\mathbb{R}^\sigma} \mathbb{R}^k \times \mathbb{R}^m \xrightarrow{f} \mathbb{R}^n \xrightarrow{\mathbb{R}^\tau} \mathbb{R}^n, r)$$

A dynamic PROP for deep learning

$$1 \to S_{1,1}, \quad 1 \to S_{0,0}, \quad \mathbf{Perm}(m) \times \mathbf{Perm}(n) \times S_{m,n} \to S_{m,n}$$

$$S_{\ell,m} \times S_{m,n} \to S_{\ell,n}, \quad \text{and} \quad S_{m_1,n_1} \times S_{m_2,n_2} \to S_{m_1+m_2,n_1+n_2}$$

- $S_{m,n} =$
$$\{(k \in \mathbb{N}, f \colon \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^n, r \in \mathbb{R}^k) | f \text{ is differentiable}\}$$
- $(0, \mathrm{id}_{\mathbb{R}}, *)$, $(0, \mathrm{id}_*, *)$
- $(\sigma, \tau, (k, f, r)) \mapsto$
$$(k, \mathbb{R}^k \times \mathbb{R}^m \xrightarrow{\mathrm{id} \times \mathbb{R}^\sigma} \mathbb{R}^k \times \mathbb{R}^m \xrightarrow{f} \mathbb{R}^n \xrightarrow{\mathbb{R}^\tau} \mathbb{R}^n, r)$$
- $((k', f', r'), (k, f, r)) \mapsto$
$$(k + k', \mathbb{R}^k \times \mathbb{R}^{k'} \times \mathbb{R}^\ell \xrightarrow{\mathrm{id} \times f'} \mathbb{R}^k \times \mathbb{R}^m \xrightarrow{f} \mathbb{R}^n, (r, r'))$$

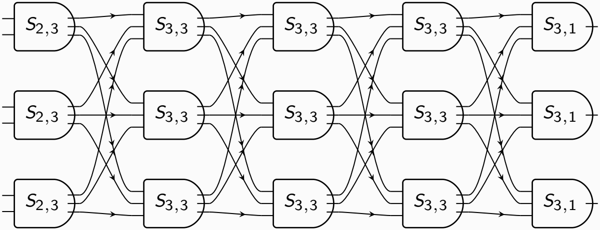
A dynamic PROP for deep learning

$$1 \to S_{1,1}, \quad 1 \to S_{0,0}, \quad \mathbf{Perm}(m) \times \mathbf{Perm}(n) \times S_{m,n} \to S_{m,n}$$

$$S_{\ell,m} \times S_{m,n} \to S_{\ell,n}, \quad \text{and} \quad S_{m_1,n_1} \times S_{m_2,n_2} \to S_{m_1+m_2,n_1+n_2}$$

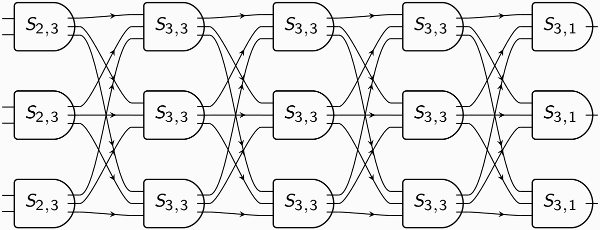
- $S_{m,n} =$
$$\{(k \in \mathbb{N}, f\colon \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^n, r \in \mathbb{R}^k) | f \text{ is differentiable}\}$$
- $(0, \mathrm{id}_{\mathbb{R}}, *)$, $(0, \mathrm{id}_*, *)$
- $(\sigma, \tau, (k, f, r)) \mapsto$
$$(k, \mathbb{R}^k \times \mathbb{R}^m \xrightarrow{\mathrm{id}\times\mathbb{R}^\sigma} \mathbb{R}^k \times \mathbb{R}^m \xrightarrow{f} \mathbb{R}^n \xrightarrow{\mathbb{R}^\tau} \mathbb{R}^n, r)$$
- $((k', f', r'), (k, f, r)) \mapsto$
$$(k + k', \mathbb{R}^k \times \mathbb{R}^{k'} \times \mathbb{R}^\ell \xrightarrow{\mathrm{id}\times f'} \mathbb{R}^k \times \mathbb{R}^m \xrightarrow{f} \mathbb{R}^n, (r, r'))$$
- $((k_1, f_1, r_1), (k_2, f_2, r_2)) \mapsto (k_1 + k_2, f_1 \times f_2, (r_1, r_2))$

Vistas



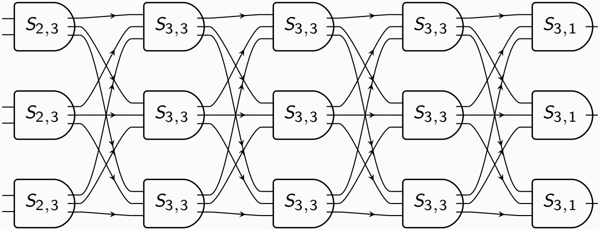
Vistas

- Model other types of learning systems compositionally as dynamic categorical structures



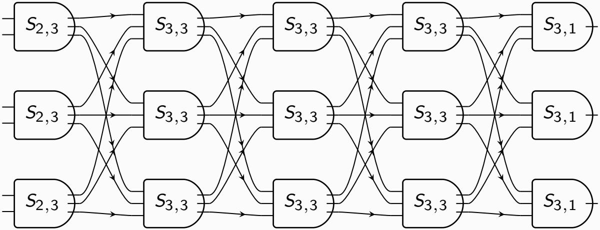
Vistas

- Model other types of learning systems compositionally as dynamic categorical structures
- As a well behaved categorical structure, dynamic PROPS can be easily compared, combined, or generalized



Vistas

- Model other types of learning systems compositionally as dynamic categorical structures
- As a well behaved categorical structure, dynamic PROPS can be easily compared, combined, or generalized
- Implementation in computing frameworks based on category theory or polynomials (algebraicjulia.org)



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Thanks!