Cusp Density: Dense or Knot?

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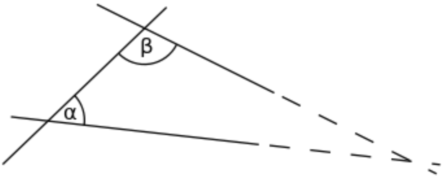
Olivetti Club 10/3/17

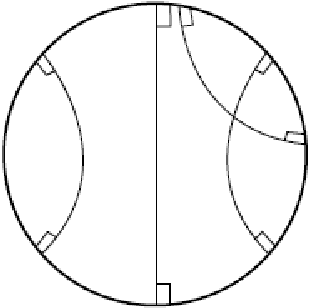
Hyperbolic Geometry

- Axiom of Euclid: *If a straight line c intersects two other straight lines a and b and makes with them two interior angles on the same side whose sum is less than two right angles, then a and b meet on that side of c on which the angles lie.*



Hyperbolic Geometry

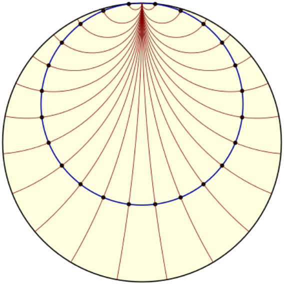
- One model for hyperbolic geometry is the Poincare disk with metric $g = \frac{2g_{Euc}}{(1-|\vec{x}|^2)^2}$, where straight lines are modeled as lines and circle arcs perpendicular to the boundary.



- Higher dimensional disks (like the unit ball in $\mathbb{R}^3$) similarly model higher dimensional hyperbolic space.

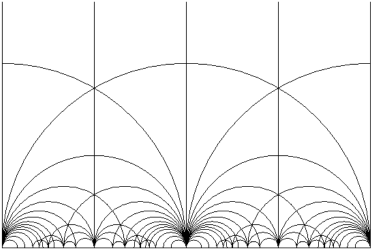
Hyperbolic Geometry

- The hyperbolic plane has a greater variety of isometries than the Euclidean plane.
- Parabolic isometries fix only a point on the boundary.
- A horocycle is the orbit of a point under the set of isometries fixing only a particular boundary point.



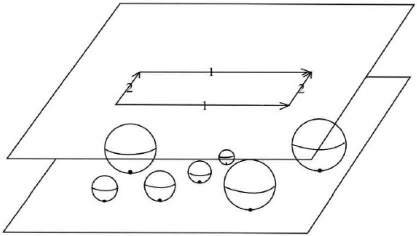
Hyperbolic Geometry

- Another model of hyperbolic space is the upper half-space, with metric $g = \frac{g_{Euc}}{z^2}$.
- Geodesics vertical lines or circles perpendicular to boundary.
- Similar to Poincare model, but with boundary flattened (except point at infinity).



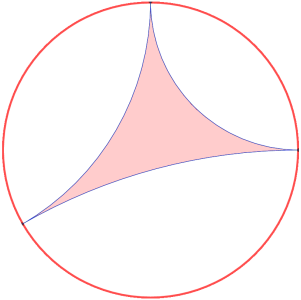
Hyperbolic Geometry

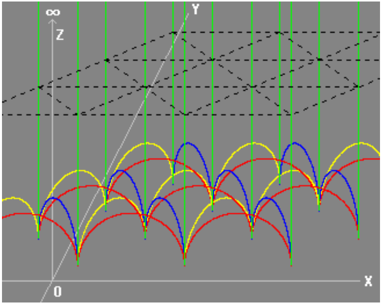
- Horospheres ‘centered’ on the xy-plane look like those in the Poincare model.
- Horospheres ‘centered’ at infinity are flat horizontal planes.
- These planes inherit a Euclidean metric, thus so do all horospheres(!)



Hyperbolic Geometry

- Polyhedra can have ‘ideal’ vertices at infinity.
- The faces of polyhedra are totally geodesic planes





Hyperbolic Geometry

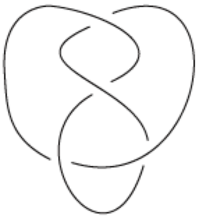
- A hyperbolic 3-manifold is a 3-manifold with a Riemannian metric having constant curvature -1.
- The volume of a hyperbolic 3-manifold is the integral of the volume form given by the metric over the entire manifold.
- By the Mostow Rigidity Theorem, a hyperbolic metric on a manifold is unique up to isometry, so hyperbolic volume is a manifold invariant.

Hyperbolic Geometry

- Any hyperbolic 3-manifold is the quotient of hyperbolic 3-space by the orbits of a discrete group of fixed point free isometries.
- Such a manifold has finite volume if a fundamental domain of that discrete group of isometries is made up of finitely many hyperbolic tetrahedra.

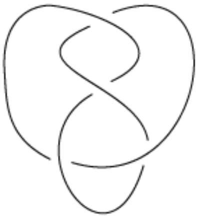
Hyperbolic Links

- A knot is a smooth embedding of S^1 in S^3.
- A link is a smooth embedding of the disjoint union of any number of copies of S^1 in S^3.
- A link is hyperbolic if the complement of its image in S^3 is a hyperbolic manifold.



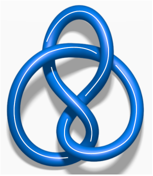
Hyperbolic Links

- The figure 8 knot has volume $2v_t$, where v_t is the volume of an ideal regular tetrahedron.
- The minimally twisted 5-chain has volume $10v_t$.
- Computing these hyperbolic structures is hard, but it can be done by a computer (SnapPea/SnapPy)



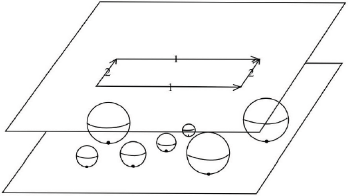
Hyperbolic Links

- A cusp in a hyperbolic 3-manifold is a $T^2 \times [0, \infty)$ neighborhood of a boundary component.
- For hyperbolic links, a cusp is a solid torus neighborhood of a component, intersected with the complement.
- A cusp is maximal when it is tangent to itself and thus cannot be further expanded.
- The cusp volume of a manifold is the maximal total volume among all configurations of nonintersecting cusps around its boundary components.



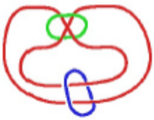
Hyperbolic Links

- A cusp in a hyperbolic manifold lifts to a disjoint union of horoballs in hyperbolic space, all of which are identified in the manifold by the covering transformations.
- Cusps arise from quotients of hyperbolic space by a group generated by two parabolic isometries about the same boundary point.
- Letting that boundary point be the point at infinity and considering the horoball centered at infinity helps make clear why this gives a tolid torus (sans core curve).

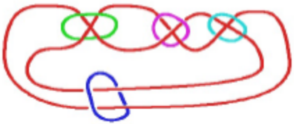


Covers and Gluings

- An n-fold cyclic cover of a link, unwinding around some component, is the link formed by cutting the complement open along a (punctured) surface bounded by the link and gluing together n copies of the manifold in a cycle.
- This can also be done for knots and general 3-manifolds, but the cover may not be a knot complement.
- Taking an n-fold cover multiplies both volume and cusp volume by n.



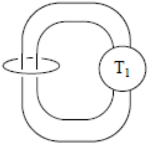
Twisted Borromean Rings



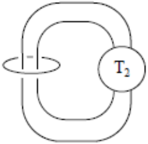
3-fold cover

Covers and Gluings

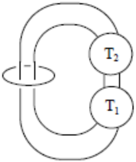
- The belted sum of two links is formed by cutting each open along a twice-punctured disk and gluing them together.
- The resulting link complement is a hyperbolic manifold as any twice-punctured disk in a hyperbolic manifold is isotopic to a totally geodesic surface with unique hyperbolic structure (Adams).
- The volume of the sum is the sum of the volumes, but the cusp volumes may not add.



L_1

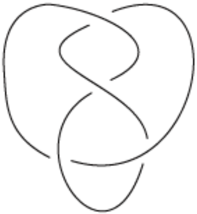


L_2



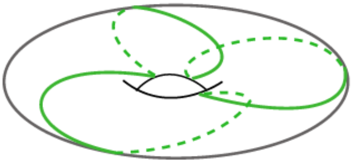
Belted Sum: L_{1+2}

- There are special cases where cusp volumes do add in belted sums.
- Any belted sum of ‘tetrahedral’ manifolds built only out of ideal regular tetrahedra has cusp volume the sum of those of its summands.
- The figure 8 knot and the minimally twisted 5-chain are tetrahedral.

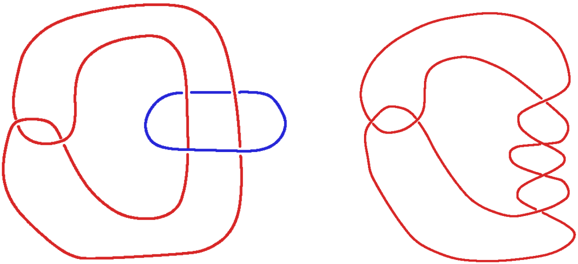


Dehn Filling

- (p,q) Dehn filling on a torus shaped boundary component of a manifold is the operation of attaching a solid torus to the manifold by gluing the meridian of the boundary of the solid torus to a (p,q) curve on the manifold boundary component.

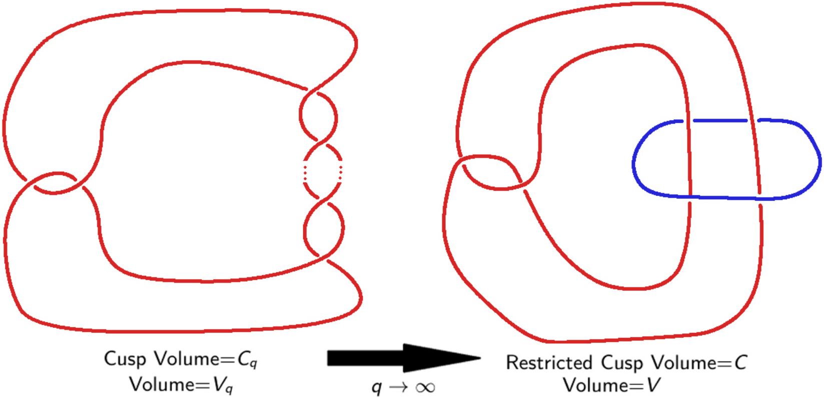


- (1,q) Dehn filling on an unknotted component of a hyperbolic link complement gives the complement of the link with the filled component removed and q full twists applied to the strands passing through it.



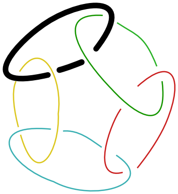
Dehn Filling

- As q approaches infinity, if a component of a hyperbolic link L is $(1, q)$ Dehn filled, the volume of the resulting manifold and the cusp volumes of the remaining components approach their original values in the complement of L.



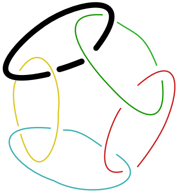
Cusp Density

- The *Cusp Density* of a hyperbolic 3-manifold is the ratio of the cusp volume to the volume of the manifold.
- The *Restricted Cusp Density* of a subset of the cusps of a manifold is the ratio of the cusp volume from just those cusps to the volume of the manifold.
- Results on horosphere packing in hyperbolic space show that cusp density is bounded above by $.853... = \frac{\sqrt{3}}{2v_t}$.



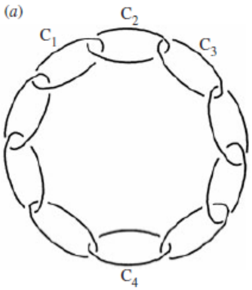
Cusp Density

- All tetrahedral manifolds have cusp density .853...
- The minimally twisted 5-chain then has cusp volume .853...
- The restricted cusp density of just one cusp is .68...



Cusp Density

- D_n is the alternating daisy chain with n components
- It is always possible to have volume of at least $\sqrt{3}/4$ in all cusps of a manifold at once, thus the the total volume of D_n goes to infinity.
- The maximal cusp volume of a single cusp approaches that of a component of the borromean rings, which is 4, so as n goes to ∞ the restricted cusp density of one cusp approaches zero.

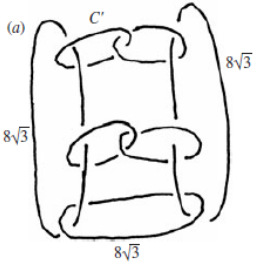


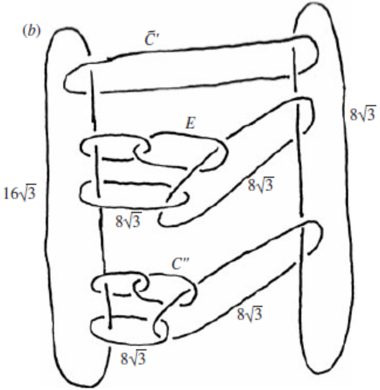
Density Construction for Manifolds

- Theorem (Adams 2001): The set of values of cusp density for finite-volume hyperbolic 3-manifolds is dense in the interval [0, .853...].
- To prove this, choose any $x \in [0, .853...]$ and construct a sequence of 3-manifolds with cusp density approaching x.

Density Construction for Manifolds

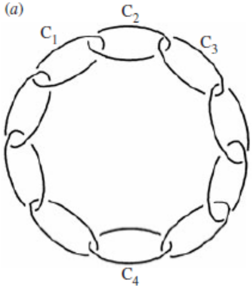
- The minimally twisted 5-chain has cusp density .853... with up to $4\sqrt{3}$ volume per cusp.
- A 2-fold cyclic cover L of the 5-chain has the same cusp density.
- Let L_k be a k-fold cyclic cover of L about the component C'

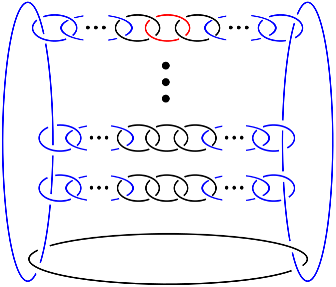




Density Construction for Manifolds

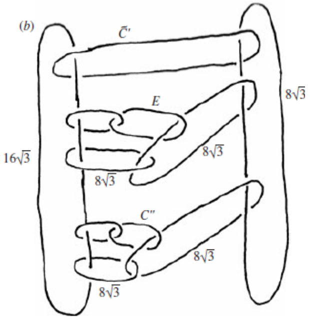
- n can be chosen so that the restricted cusp density of the labelled components is arbitrarily close to 0.
- The same then holds for the lifts of those components in the m-fold cyclic cover $D_{n,m}$

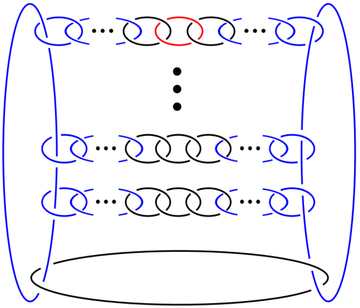




Density Construction for Manifolds

- Define $F_{k,n,m}$ as the belted sum of L_k and $D_{n,m}$ along the highlighted disk and E.
- Choose even n large enough so that the maximal volume in each cusp is within .1 of 4 and the restricted cusp density of the cusps $C_1, ..., C_4$ is less than x.
- The cusp volumes add, up to a constant $p < 16\sqrt{3} + 12.3$.



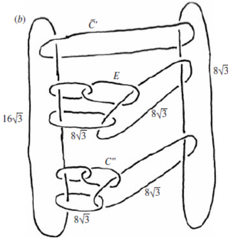


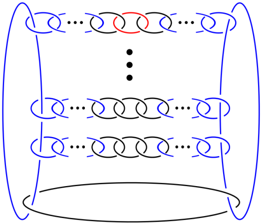
Density Construction for Manifolds

- The cusp density of $F_{k,n,m}$ restricted to the cusps in L_k and those in black in $D_{n,m}$ is then, in terms of the volumes V_L, V_D of L and D_n and cusp volumes CV_L, CV_D, is

$$\frac{kCV_L + mCV_D - p}{kV_L + mV_D} = \frac{\frac{k}{m}CV_L + CV_D}{\frac{k}{m}V_L + V_D} - \frac{p}{kV_L + mV_D}$$

- k and m can be made arbitrarily large, making the second term negligible without affecting the first.



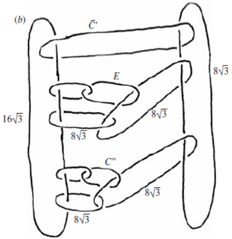


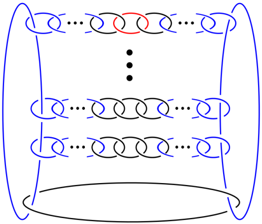
Density Construction for Manifolds

- Replace $\frac{k}{m}$ with a real variable t:

$$f(t) = \frac{tCV_L + CV_D}{tV_L + V_D} - \epsilon$$

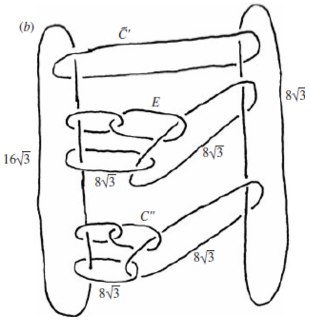
- As t goes to 0, $f(t)$ goes to the restricted cusp density of D_n (less than x), and as t goes to infinity $f(t)$ goes to .853...
- Thus t can be chosen such that $f(t) = x$, and as f is continuous this value can be approached by the images of a sequence of rationals approaching t.

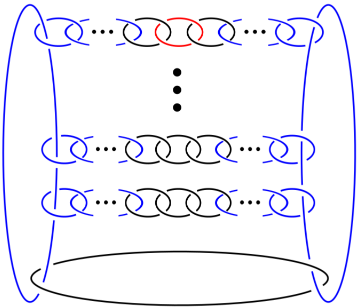




Density Construction for Manifolds

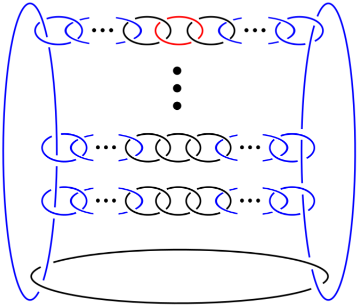
- We now have a means of constructing manifolds with a restricted cusp density arbitrarily close to x.
- Now just do $(1, q)$ Dehn filling on each of the remaining (blue) components, which for high enough q has volume and cusp volumes arbitrarily close to those in the original manifold.
- The filled manifolds now have cusp density arbitrarily close to x, completing the proof.





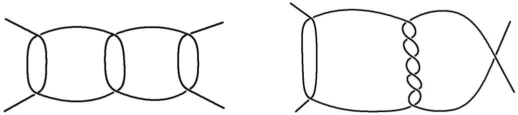
Density Construction for Links

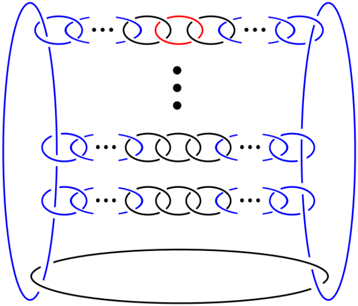
- This construction shows that the cusp densities of finite volume hyperbolic 3-manifolds are dense in [0,.853...].
- However, it is not hard to show that the manifolds constructed are link complements.
- This would then show that the cusp densities of link complements are dense in the same interval.



Density Construction for Links

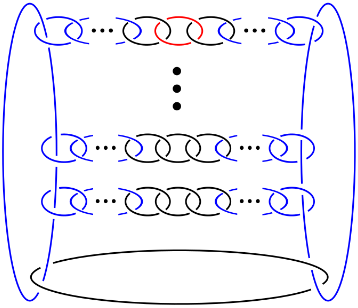
- Use result on Dehn filling an unknotted component:



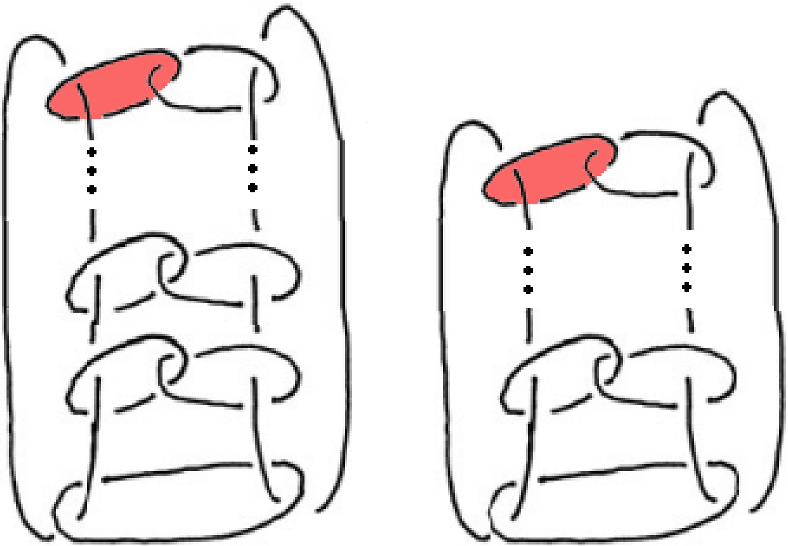


Density Construction for Links

- This shows the manifold obtained by filling on the blue components below is a link complement.
- The same argument applies whenever the unknotted components have no cycles in their adjacencies.



Density Construction for 1-Cusped Manifolds



Directions

- Find examples to extend result to entire interval for 1-cusped manifolds.
- Do the same for knots (harder).
- Find sequences of knots with cusp density approaching .853...

Acknowledgements

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- SnapPy

References

1 Colin Adams (2002). "Cusp Densities of Hyperbolic 3-Manifolds" *Proceedings of the Edinburgh Mathematical Society* **45**, 277-284

2 W. Thurston (1978). "The geometry and topology of 3-manifolds", *Princeton University lecture notes (http://www.msri.org/gt3m)*.

3 R. Meyerhoff (1978). "Geometric Invariants for 3-Manifolds" *The Mathematical Intelligencer* **14** 37-52.